

Feel free to ask me questions!

Here's what you need to know to get the perfect grade.

critical pt. local max min.

(1) 5.6 Optimization: Be familiar with these two problems.

(a) Example: Suzie can sell 20 bracelets each day when the price is \$10 for a bracelet. If she raises the price by \$1, then she sells 2 fewer bracelets each day. If it costs \$8 to make each bracelet, find the selling price that will maximize Suzie's profit.

If x is critical pt. of $f(x)$, and $f''(x) < 0 \Rightarrow f(x)$ is local max.
 $f''(x) > 0 \Rightarrow f(x)$ is local min.
 $R''(p) = -4 < 0$
 \Rightarrow

maximize Suzie's profit.

$$R(x) = (\text{price} - \text{cost}) \cdot \text{quantity} = \text{price} \cdot \text{quantity} - \text{total cost.}$$

$$R(p) = (p - 8)Q(p)$$

$$= (p - 8)(-2p + 40)$$

$$= -2p^2 + 16p + 40p - 320$$

dom $R(p) = [0, \infty)$

$$R'(x) = -4p + 56$$

Critical values.

$$R'(x) = 0 \text{ or DNE.}$$

$$-4p + 56 = 0 \Rightarrow 4p = 56 \Rightarrow p = 14 \leftarrow \text{critical value.}$$

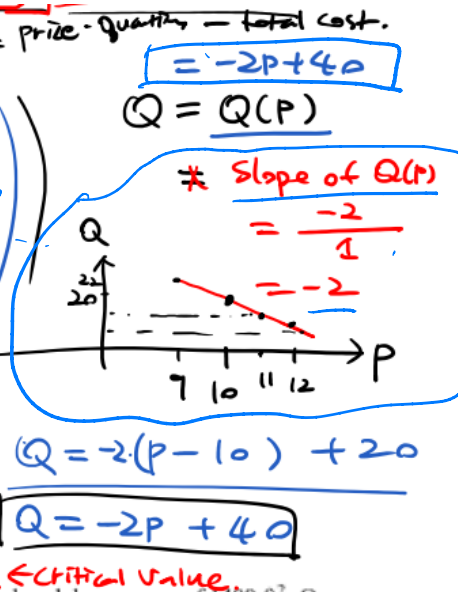
Q	P
20	\$10
18	\$11
16	\$12

$$R(14) = (14 - 8)(-2(14) + 40)$$

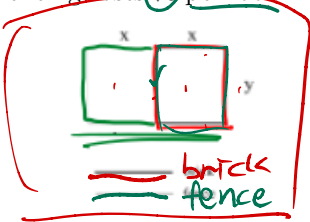
$$= 6 \cdot 12 = 72 \$$$

$$R(0) = 0$$

Selling price = 14\$



(b) Example: Ben needs to enclose two rectangular regions that share one side, and each has an area of 1400 ft². One of the regions needs to be enclosed on all four sides by a brick wall, and the other region just needs fencing on its remaining three sides (see diagram below). The brick wall costs \$18 per foot, and the fencing costs \$6 per foot. Find the dimensions of each region that would be the most economical for Ben.



$$x = 35 \sqrt{2}$$

$$y = \frac{1400}{2 \cdot 35} = \dots$$

$$C(x) = (2x + 2y) \cdot 18 + (2x + y) \cdot 6$$

Minimize $C(x)$.

s.t. $y = 1400 \text{ ft}^2, y = \frac{1400}{x}$

$$C(x) = (2x + 2 \cdot \frac{1400}{x}) \cdot 18 + (2x + \frac{1400}{x}) \cdot 6$$

$$= 36x + \frac{1400 \cdot 36}{x} + 12x + \frac{6 \cdot 1400}{x}$$

$$= 48x + \frac{1400}{x} (36 + 6)$$

$$= 48x + \frac{42 \cdot 1400}{x}$$

Minimize $C(x) = 48x + \frac{42 \cdot 1400}{x}$

s.t. $x \geq 0, (0, \infty)$

$$\Rightarrow C'(x) = 48 - 42 \cdot 1400 \cdot x^{-2}$$

$$x^2 = \frac{1}{x^2}$$

Find x where $C'(x) = 0$ or DNE \Rightarrow

$$48 - (42 \cdot 1400 - x^2) = 0$$

$$48x^2 = 42 \cdot 1400$$

$$x^2 = \frac{42 \cdot 1400}{48}$$

$$x = \pm \sqrt{\frac{42 \cdot 1400}{48}}$$

$$= \pm \frac{\sqrt{3 \cdot 140}}{\sqrt{12} \cdot 4} = \pm \frac{140}{4} = \pm 35$$

$$\sqrt{48} = \sqrt{3 \cdot 16} = \sqrt{3} \cdot \sqrt{16} = 4\sqrt{3}$$

$$42 = 3 \cdot 7 \cdot 2 \quad 48 = 2^4 \cdot 3$$

$$140 = 2 \cdot 7 \cdot 10 \cdot 10$$

$$42 \cdot 1400 = 3 \cdot 2^2 \cdot 7^2 \cdot 10^2$$

$$\sqrt{\quad} = \sqrt{3 \cdot 2 \cdot 7 \cdot 10}$$

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5-6 problems

1. Find an equation.

(1) It is function of price on quantity...

Revenue = (price - cost) # of product

product = function of price

(2) two variables x, y but there is some relationship between x, y .

Critical value = +35

Because -35 is not in the domain

$$C''(35) = 84 - 1400 \cdot \frac{1}{(\frac{35}{\sqrt{3}})^3} > 0$$

\Rightarrow By second derivative test, C has local minimum on $x = 35$

$$C'(x) = -21 \cdot 1400 \cdot x^{-3} (-2)$$

$$= 42 \cdot 1400 x^{-3}$$

\Rightarrow Since $x = 35$ is the only critical value and there is no boundary points, it is absolute minimum

$$x = 35 \quad y = \frac{1400}{35} = 40$$

(2) 6.1 Antiderivatives

- (a) **Antiderivative** of a function $f(x)$: a function $F(x)$ such that $F'(x) = f(x)$.
- (b) **Indefinite integral**: $\int f(x) dx = F(x) + C$ such that C is constant, F is an antiderivative.
- (c) Properties of **indefinite integral**: For constant C and k ,

- (i) $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$, where $n \neq -1$
- (ii) $\int k dx = kx + C$
- (iii) $\int e^x dx = e^x + C$
- (iv) $\int \frac{1}{x} dx = \ln|x| + C$, where $x \neq 0$
- (v) $\int kf(x) dx = k \int f(x) dx$
- (vi) $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

ex 2

$$C(x, y)$$

$$xy = 1400$$

$$y = \frac{1400}{x}$$

2. Find critical values

3. Derivative second derivative

4. Second derivative test + conclude local max or local min.

$$\left(\ln |x| \right)' = \frac{1}{x} \quad n = -1$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

(d) Example 2, i): Find an indefinite integral $\int \frac{4u+u^3-3u^{-7}}{5u^2} du$

$$= \int \left(\frac{4u}{5u^2} + \frac{u^3}{5u^2} - \frac{3u^{-7}}{5u^2} \right) du = \int \left(\frac{4}{5} u^{-1} + \frac{1}{5} u - \frac{3}{5} u^{-9} \right) du$$

$$\stackrel{(6)}{=} \int \frac{4}{5} u^{-1} du + \int \frac{1}{5} u du + \int \left(-\frac{3}{5} \right) u^{-9} du \stackrel{(5)}{=} \frac{4}{5} \int u^{-1} du + \frac{1}{5} \int u du - \frac{3}{5} \int u^{-9} du$$

$$= \frac{4}{5} (\ln |u| + C) + \frac{1}{5} (u^2 + C) - \frac{3}{5} \frac{1}{-9+1} u^{-8}$$

$$= \frac{4}{5} \ln |u| + \frac{1}{5} u^2 + \frac{3}{40} u^{-8} + C$$

(e) Example 4: The marginal revenue of selling x watches each day is given by $R'(x) = 30 - 0.0003x^2$ dollars per watch for $0 \leq x \leq 540$. If the revenue is \$1487.50 when 50 watches are sold, find the revenue function.

$$R(50) = 1487.50$$

$$R(x) = 30x - 0.0001x^3$$

$$(1) \int R'(x) dx = \int (30 - 0.0003x^2) dx = \int 30 dx + \int -0.0003x^2 dx$$

$$= 30x + C - 0.0003 \int x^2 dx$$

$$= 30x + C - 0.0003 \left(\frac{1}{3} x^3 + C \right)$$

① Find an antiderivative using indefinite integral

② Plus in the given value of $R(x)$

plus in
 a) 50 into $Y(x) = 30x - 0.0001x^3 + C$
 $1487.5 = Y(50) = 30 \cdot 50 - 0.0001 \cdot 50^3 + C = 1487.50$

Example 5: A sculpture purchased by a museum for \$50,000 increases in value at a rate of $V'(t) = 100e^t$ dollars $\Rightarrow C = 0$

(f) Also see Example 5 in the previous lecture notes.

(3) 6.2: Substitution

(a) Reversing the chain rule!

(b) Example 1: $\int e^{x^3-1} \cdot 3x^2 dx = \int e^u du = e^u + C$
 $u = x^3 - 1$
 $du = 3x^2 \cdot dx$
 $= e^{x^3-1} + C$

$du = f'(x) dx$

(c) General Indefinite Integral Formulas

(i) $\int (f(x))^n \cdot f'(x) dx = \frac{(f(x))^{n+1}}{n+1} + C, n \neq -1$

Ex 1 (ii) $\int e^{f(x)} \cdot f'(x) dx = e^{f(x)} + C$

(iii) $\int \frac{1}{f(x)} \cdot f'(x) dx = \ln |f(x)| + C$

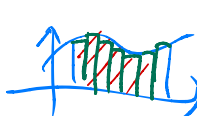
(d) Example 3 c): $\int \frac{2e^{5/x^4}}{3x^5} dx$

(ii) $\int \frac{2e^{5/x^4}}{3x^5} dx = \int e^{u} \cdot \frac{2}{3} dx$
 $u = \frac{5}{x^4} = 5 \cdot x^{-4}$
 $\frac{du}{dx} = (-4) \cdot 5 \cdot x^{-4-1} = -20 \cdot x^{-5}$
 $du = -20 x^{-5} \cdot dx$
 $\frac{1}{-20} du = x^{-5} dx$
 $\frac{1}{20} \cdot dx$
 $\int \frac{2}{3} e^u \cdot \frac{1}{20} du = \frac{2}{3} \int e^u \cdot \frac{1}{20} du$
 $= \frac{2}{3} \cdot \frac{1}{20} \cdot e^u + C = \frac{1}{30} e^u + C$
 $= \frac{1}{30} e^{5/x^4} + C$

(e) Example 3 d) $\int \frac{8t^3}{\sqrt{2-5t^4}} dt = \int 8t^3 \cdot (2-5t^4)^{-1/2} dt$
 $u = 2-5t^4$
 $\frac{du}{dt} = -5 \cdot 4 t^3 = -20 t^3$
 $du = -20 t^3 dt$
 $-\frac{1}{20} du = t^3 dt$
 $\int 8 \int u^{-1/2} \cdot \left(-\frac{1}{20}\right) du$
 $= \frac{8}{-20} \int u^{-1/2} du$

$$\begin{aligned}
 \int \frac{8t^3}{\sqrt{2-5t^4}} dt &= 8 \int \frac{t^3}{\sqrt{2-5t^4}} dt = 8 \int \frac{-\frac{1}{20}}{u^{\frac{7}{4}}} du \\
 &= \frac{-8}{20} \int \frac{1}{u^{\frac{7}{4}}} du = \frac{-2}{5} \int u^{-\frac{7}{4}} du \\
 &= \frac{-2}{5} \left(\frac{1}{-\frac{7}{4}+1} \cdot u^{-\frac{7}{4}+1} + C \right) \\
 &= \frac{-2}{5} \left(\frac{1}{-\frac{3}{4}} \cdot u^{-\frac{3}{4}} + C \right) = \frac{-2}{5} \left(-\frac{4}{3} u^{-\frac{3}{4}} + C \right) \\
 &= \frac{8}{15} \cdot u^{-\frac{3}{4}} + C = \frac{8}{15} \cdot (2-5t^4)^{-\frac{3}{4}} + C
 \end{aligned}$$

(4) 6.3: Estimating Distance Traveled



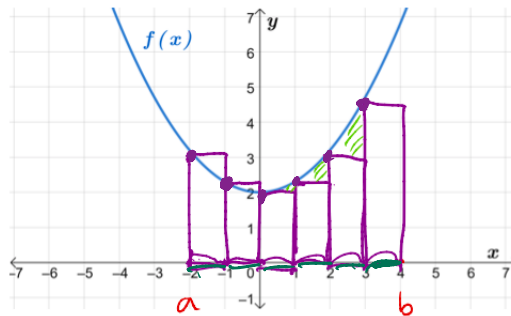
(a) We will estimate the area under a curve from $x = a$ to $x = b$ by dividing the region into subintervals (rectangles) of equal width.

$$\text{width of each subinterval} = \Delta x = \frac{b-a}{n}$$

where n is the number of subintervals (rectangles).

(b) (Left Sum) Example 1: For the function $f(x) = 0.3x^2 + 2$, estimate the area of the region that lies under the graph of $f(x)$ between $x = -2$ to $x = 4$ using a left-hand sum with six subintervals of equal width.

Example 1: For the function $f(x) = 0.3x^2 + 2$, estimate the area of the region that lies under the graph of $f(x)$ between $x = -2$ to $x = 4$ using a left-hand sum with six subintervals of equal width.



$$n = 6$$

$$b = 4$$

$$a = -2$$

$$\Delta x = \frac{4 - (-2)}{6} = \frac{6}{6} = 1$$

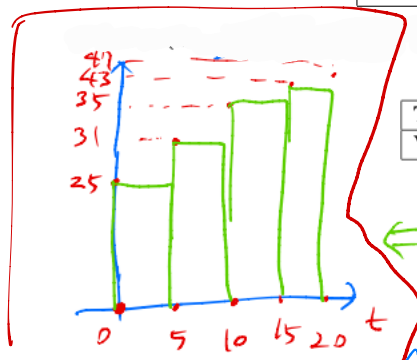
$$L_n = f(-2) \cdot 1 + f(-1) \cdot 1 + f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 = 17.7$$

\Rightarrow Underestimate

Sum of all rectangles

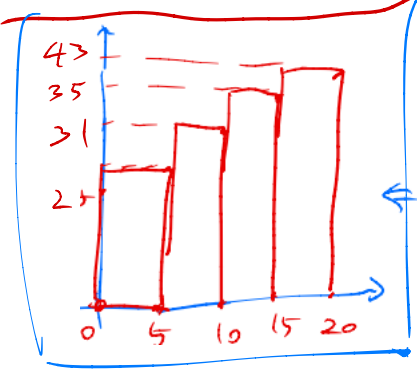
(c) Example 6: The table below shows the velocity (ft/s) of an object every five seconds over a 20 second time interval. Estimate the total distance the object travels over the 20 second time interval by finding upper and lower estimates (i.e., right and left sums).

Time (s)	0	5	10	15	20
Velocity (fts)	25	31	35	43	47



Time (s)	0	5	10	15	20
Velocity (ft/s)	25	31	35	43	47

Left Sum $L_4 = v(0) \cdot 5 + v(5) \cdot 5 + v(10) \cdot 5 + v(15) \cdot 5 = 25 \cdot 5 + 31 \cdot 5 + 35 \cdot 5 + 43 \cdot 5 = 670 \text{ ft}$



Right Sum $R_4 = v(5) \cdot 5 + v(10) \cdot 5 + v(15) \cdot 5 + v(20) \cdot 5 = 31 \cdot 5 + 35 \cdot 5 + 43 \cdot 5 + 47 \cdot 5 = 980 \text{ ft}$

(5) 6.4: The Definite Integral

Rightmost, left most, up

- (a) In general, we can use any x -coordinate, x_i^* , to find the height of the rectangle in the i^{th} subinterval. Using summation notation, we can write the sum of the areas of the rectangles as

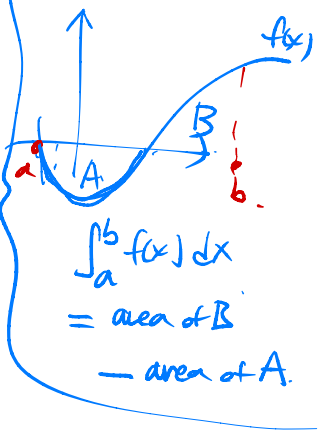
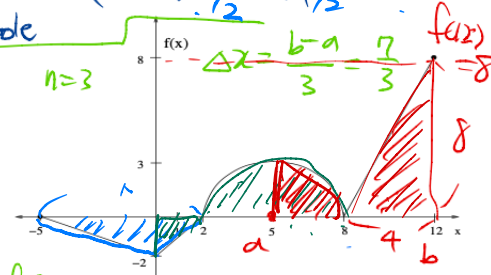
$$f(x_1^*) \Delta x + f(x_2^*) \Delta x + \dots + f(x_n^*) \Delta x = \sum_{i=1}^n f(x_i^*) \Delta x = \text{approximate}$$

The sum $\sum_{i=1}^n f(x_i^*) \Delta x$ is called a Riemann sum. *area of rectangles area between f(x) and x-axis*

- (b) Then, the **definite integral** of $f(x)$ from a to b is $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$
 (c) Example 1: Use the graph of $f(x)$ below to find the following. Note that the graph consists of three straight lines and a semicircle.

a) $\int_{-5}^2 f(x) dx$
 In $x \in (-5, 2)$, $f(x) < 0$
 $= - (b-a) \cdot \text{height of triangle} = - (2 - (-5)) \cdot \frac{2}{2} = -14/2 = -7$

$f(x) = \begin{cases} -\frac{2}{3}x - 2 & -5 < x < 0 \\ x - 2 & 0 \leq x < 2 \end{cases}$
 $n=3$



b) $\int_0^8 f(x) dx$
 = Area of semicircle - Area of triangle
 $= \pi \cdot 3^2 \cdot \frac{1}{2} - 2 \cdot 2 \cdot \frac{1}{2}$
 $= \frac{9}{2} \pi - 2$

$f(-5 + \frac{2}{3}) \cdot \frac{7}{3} + f(-5 + \frac{4}{3}) \cdot \frac{7}{3}$
 ≈ -14

c) $\int_5^{12} f(x) dx$
 = Area of quarter of the circle + Area of triangle
 $= \pi \cdot 3^2 \cdot \frac{1}{4} + 4 \cdot 8 \cdot \frac{1}{2}$
 $= \frac{9}{4} \pi + 16$

(d) Example 3: Use a midpoint sum with $n = 3$ to estimate $\int_{-1}^2 (x^2 - 1) dx$

Example 3: Use a midpoint sum with $n = 3$ to estimate $\int_{-1}^2 (x^2 - 1) dx$. $b = 2$, $a = -1$

$f(x) = x^2 - 1$
 $f(-1) = (-1)^2 - 1 = 1 - 1 = 0$
 $f(2) = 2^2 - 1 = 3$
 $f(0) = -1$

$\Delta x = \frac{2 - (-1)}{3} = \frac{3}{3} = 1$

$f(0.5) = f(-0.5) = 0.25 - 1 = -0.75$
 $f(1.5) = 2.25 - 1 = 1.25$

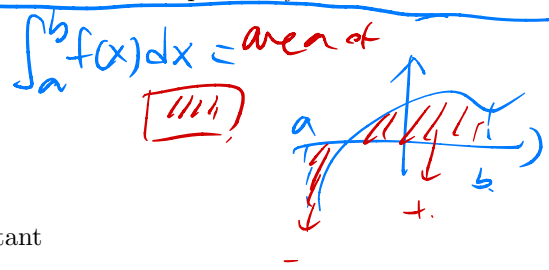
$-2 \times 0.75 + 1 \times 1.25 = -1.5 + 1.25 = -0.25 \approx \int_{-1}^2 (x^2 - 1) dx$

(6) 6.5. The Fundamental Theorem of Calculus

(a) $\int_a^b f(x) dx$ gives an exact value and "counts" area above the x -axis positively and area below the x -axis negatively.

(b) Properties of Definite Integral:

- (i) $\int_a^b m dx = m(b - a)$, where m is a constant
- (ii) $\int_a^a f(x) dx = 0$
- (iii) $\int_a^b f(x) dx = -\int_b^a f(x) dx$
- (iv) $\int_a^b k f(x) dx = k \int_a^b f(x) dx$, where k is a constant
- (v) $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- (vi) $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, where $a < c < b$



(c) Example 2: Use the graph of $f(x)$ with the indicated areas below to answer the following.

area of A: 2.0
 area of B: 1.5
 area of C: 2.5
 area of D: 9.5

area A = $\int_{-1}^0 f(x) dx$
 area B = $-\int_0^1 f(x) dx$
 area C = $-\int_1^2 f(x) dx$
 area D = $\int_2^3 f(x) dx$

a) Find $\int_a^c f(x) dx - \int_0^c 2f(x) dx$

$\int_a^c f(x) dx = \text{area A} - \text{area B} - \text{area C} = 2.0 - 1.5 - 2.5 = -2.0$

$\int_0^c 2f(x) dx = 2 \int_0^c f(x) dx = 2(\text{area B} + \text{area C}) = 2(1.5 + 2.5) = 8.0$

$\int_a^c f(x) dx - \int_0^c 2f(x) dx = -2.0 - 8.0 = -10.0$

b) Find $\int_0^a 4f(x) dx + \int_a^b f(x) dx$

$\int_0^a 4f(x) dx = 4 \int_0^a f(x) dx = 4(\text{area B}) = 4(1.5) = 6.0$

$\int_a^b f(x) dx = \text{area C} = 2.5$

$\int_0^a 4f(x) dx + \int_a^b f(x) dx = 6.0 + 2.5 = 8.5$

(d) The Fundamental Theorem of Calculus, Part 2 - Suppose f is continuous on $[a, b]$

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f , that is, $F' = f$

(e) Example 5: Evaluate $\int_2^k (t^2 + 4) dt$

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Example 5: Evaluate $\int_2^k (t^2 + 4) dt = \int_2^k t^2 dt + \int_2^k 4 dt$

$$= \left. \frac{1}{3} t^3 \right|_2^k + [4 \cdot (k - 2)]$$

FTC

$$= \frac{1}{3} k^3 - \frac{1}{3} 2^3 + 4k - 8$$

$$= \frac{1}{3} k^3 - \frac{8}{3} + 4k - 8 = \frac{1}{3} k^3 + 4k - \frac{32}{3}$$

(f) Ex6:

Example 6: A honeybee population starts with 200 honeybees and increases at a rate of $n'(t) = 100e^{2t}$ bees per week, where t is in weeks and $t \geq 0$.

a) Find the change in the honeybee population over the first 4 weeks. Round to the nearest integer, if necessary

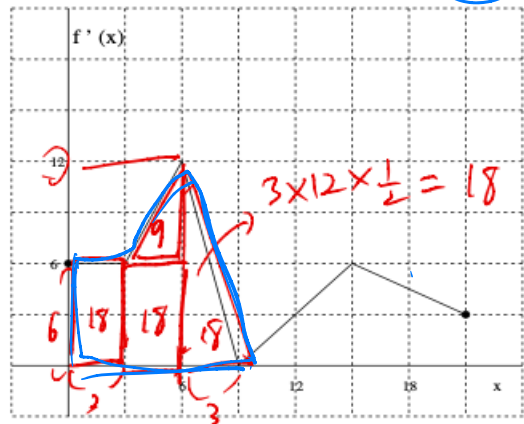
$n(4) - n(0) = \int_0^4 n'(t) dt = \int_0^4 100e^{2t} dt$

Use calculator to calculate definite integral.

$$= 148997.8994 \approx 148998 \text{ bees}$$

(g) Example 7: Consider the graph of $f'(x)$ shown below. If $f(0) = 50$, find $f(9)$.

Example 7: Consider the graph of $f'(x)$ shown below. If $f(0) = 50$, find $f(9)$.



$$f(9) - f(0) = \int_0^9 f'(x) dx$$

FTC

$$50 = 63$$

$$f(9) - 50 = 63$$

$$f(9) = 113$$

$$18 + 18 + 18 + 9 = 63$$

(h) Average Value of a Continuous Function f over $[a, b]$

$$\frac{1}{b-a} \int_a^b f(x) dx$$

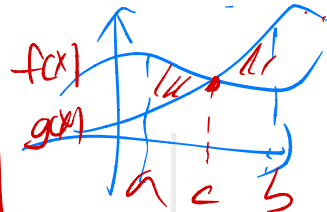
$$f(a) - f(0) = \int_0^a f'(x) dx$$

(7) 6.6: Area Between Two Curves

(a) Theorem: If $f(x)$ and $g(x)$ are two continuous functions with $f(x) \geq g(x)$ on $[a, b]$, then the area between the two curves on $[a, b]$ is given by

$$\int_a^b (f(x) - g(x)) dx.$$

(b) Example 5: Find the area that is bounded by $y = -x^2$ and $y = 2x^3 - 5x$



① Find intersection pt.

$$-x^2 = 2x^3 - 5x$$

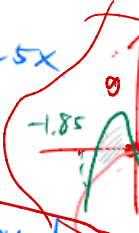
$$\Rightarrow 0 = 2x^3 + x^2 - 5x$$

$$= x(2x^2 + x - 5)$$

$$= x$$

quadratic formula to figure out intersection
calculator to figure out intersection

$$x = \frac{0 \pm \sqrt{1.3508 - 1.8508}}{2}$$



$$1 \in [0, 1.3508]$$

$$f(1) = -1$$

$$g(1) = 2 - 5 = -3$$

$$= -3$$

$$\Rightarrow f \geq g$$

$$\text{on } [0, 1.3508]$$

$$-1 \in [-1.8508, 0]$$

$$f(-1) = -1$$

$$g(-1) = -2 + 5 = 3$$

$$g \geq f \text{ on } [-1.8508, 0]$$

Area

$$= \int_a^c f - g dx$$

$$+ \int_c^b g - f dx$$

② Draw your function with intersection

③ Determine which interval $f \geq g$ or $g \geq f$.

$$\textcircled{3} \int_{-1.8508}^0 g(x) - f(x) dx = \int_{-1.8508}^0 (2x^3 - 5x + x^2) dx$$

$$+ \int_0^{1.3508} f(x) - g(x) dx = \int_0^{1.3508} (-x^2 - 2x^3 + 5x) dx$$

(c) Example 7: Set up the definite integral(s) representing the area bounded by $y = -x^2 + 10x - 17$ and the x -axis on $[5, B]$, where $B > 8$

Example 7: Set up the definite integral(s) representing the area bounded by $y = -x^2 + 10x - 17$ and the x -axis on $[5, B]$ where $B > 8$.

$$g(x) = 0 \quad f(x) = -x^2 + 10x - 17$$

quadratic formula

(calculator)

$$x = 7.8284 < 8$$

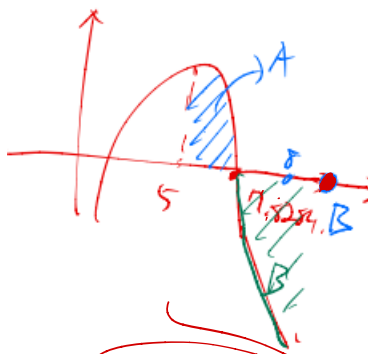
$$(2.1716)$$

$$A = \int_5^{7.8284} f(x) dx = \int_5^{7.8284} (-x^2 + 10x - 17) dx$$

$$B = \int_{7.8284}^B -f(x) dx = \int_{7.8284}^B (+x^2 - 10x + 17) dx$$

$$= \left[\frac{1}{3} x^3 - 5x^2 + 17x \right]_5^B$$

$$= \left[\frac{1}{3} B^3 - 5B^2 + 17B - \left(\frac{1}{3} (7.8284)^3 - 5(7.8284)^2 + 17(7.8284) \right) \right]$$



$$\int_5^{7.82} f(x) dx$$

$$- \int_{7.82}^B f(x) dx$$

$$= -\frac{1}{3}x^3 + 5x^2 - 17x$$

$$\frac{46}{3} + \frac{7}{2} + \frac{17}{3} + \frac{7}{2}$$

$$= 21 + 9 = 30$$