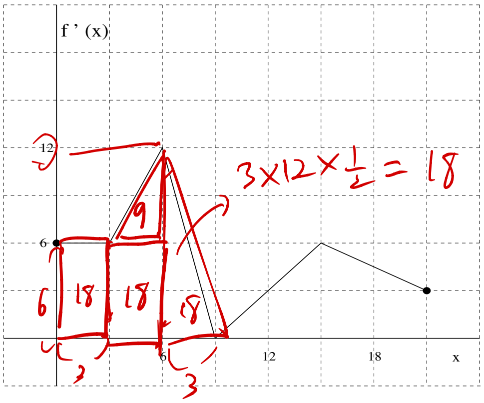
Feel free to ask Question!

Example 7: Consider the graph of $f'(x)$ shown below. If $f(0) = 50$, find $f(9)$.



$$\underbrace{f(9) - f(0)}_{50} = \underbrace{\int_0^9 f'(x)\,dx}_{63}$$ FTC

$18+18+18+9 = 63$

$f(9) - 50 = 63$

$f(9) = 113$

Average Value of a Continuous Function f over $[a,b]$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$Y_1 = \dfrac{\text{fnInt}(\sqrt{x+2}, x, 2, 7)}{5}$

$=$

f(x)

A

a b

width

Average Value of function = approximate height of the area $= \int_a^b f(x)\,dx \,/\,$ width

area = height × width

Example 8: Find the average value of $f(x) = \sqrt{x+2}$ on $[2,7]$.

$\dfrac{1}{7-2} \cdot \int_2^7 \sqrt{x+2}\,dx$

a b u

- substitution

$u = x+2$

$du = dx$ $u(2) = 2+2 = 4$

$u(7) = 7+2 = 9$

$= \dfrac{1}{5}\int_4^9 u^{\frac{1}{2}}\,du$

$\underset{FTC}{=} \dfrac{1}{5}\cdot\dfrac{1}{\frac{1}{2}+1}\cdot u^{\frac{1}{2}+1}\Big|_{u=4}^{u=9} = \dfrac{1}{5}\left(\dfrac{2}{3}\cdot 9^{\frac{3}{2}} - \dfrac{2}{3}\cdot 4^{\frac{3}{2}}\right) = \dfrac{2}{15}\cdot(27-8) = \boxed{\dfrac{38}{15}}$

Example 9: A company's marginal cost function is given by $m(x) = 0.3x^2 + 2x$ dollars per item, where x is the number of items produced. Find

↳ derivative of total cost.

a) the change in the total cost when the number of items produced increases from 10 to 20.

$C(x)$: total cost function

$\Rightarrow C'(x) = m(x)$.

$\$1000$

$C(20) - C(10) \underset{FTC}{=} \int_{10}^{20} C'(x)\,dx = \boxed{\int_{10}^{20} m(x)\,dx} = \boxed{\int_{10}^{20} 0.3x^2 + 2x\,dx}$

$= 0.3\int_{10}^{20} x^2\,dx + 2\cdot\int_{10}^{20} x\,dx$

$= 0.3\,\frac{1}{3}(20^3 - 10^3)$

$+ 2\cdot\left(\frac{1}{2}20^2 - \frac{1}{2}10^2\right)$

$= 0.1(8000 - 1000)$

$+ (400 - 100) = 700 + 300$

$= \boxed{\$1000}$

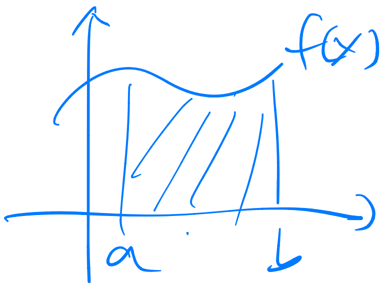
b) the average marginal cost over the interval $[10,20]$.

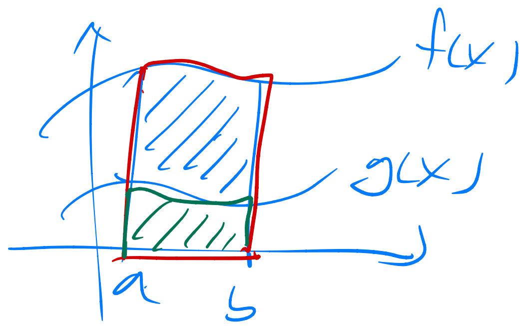
$\dfrac{1}{20-10}\cdot\int_{10}^{20} m(x)\,dx = \dfrac{1}{10}\cdot\$1000$

$= \$100./\text{unit}.$

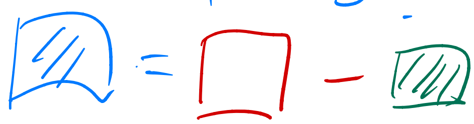
Section 6.6: Area Between Two Curves

Question: How can we use definite integrals to find the area between two continuous functions on an interval?





$f(x) \geq g(x)$

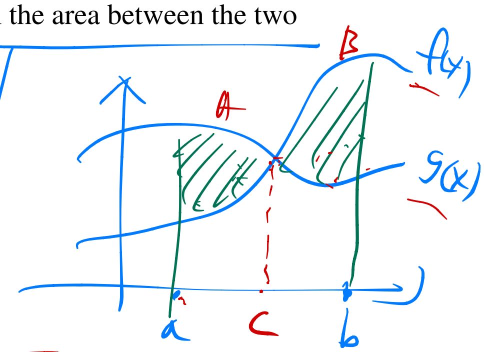


$$= \int_a^b (f(x) - g(x))\,dx$$

$$\int_a^b f(x)\,dx \qquad \int_a^b g(x)\,dx$$

Theorem: If $f(x)$ and $g(x)$ are two continuous functions with $f(x) \geq g(x)$ on $[a,b]$, then the area between the two curves on $[a,b]$ is given by

$$\int_a^b \left(f(x) - g(x)\right) dx$$



$A + B$

$$= \int_a^c (g(x) - f(x))\,dx + \int_c^b (f(x) - g(x))\,dx$$

$f(x) = x$ $\quad g(x) = \frac{1}{2}x^2 + 2$

Example 1: Find the area that is bounded by the curves $y = x$ and $y = \frac{1}{2}x^2 + 2$ on $[-4, 3]$.

① Find intersection of the function

$f(x) = x, \quad g(x) = \frac{1}{2}x^2 + 2$

$x = \frac{1}{2}x^2 + 2 \Rightarrow \frac{1}{2}x^2 - x + 2 = 0$

$\Rightarrow x^2 - 2x + 4 = 0.$

$\Rightarrow (x-1)^2 = -3$

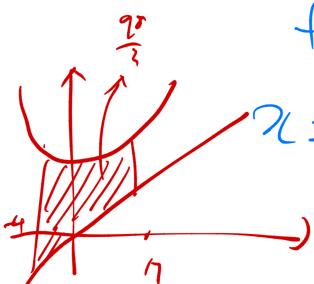
$\Rightarrow$ There is no real solution

$\Rightarrow$ No intersection have found!

② Whether $f \geq g$ or $g \geq f$ on $[-4, 3]$

$x = 1 \Rightarrow f(1) = 1, \quad g(1) = \frac{1}{2} + 2 \Rightarrow g(1) \geq f(1)$

$\Rightarrow g(x) \geq f(x)$ on $[-4, 3]$



③ Use the theorem

$$\int_{-4}^{3} (g(x) - f(x))\,dx$$

$$= \int_{-4}^{3} \left(\tfrac{1}{2}x^2 + 2 - x\right) dx$$

$= \text{fnInt}\left(\frac{1}{2}x^2 + 2 - x, x, -4, 3\right)$

$= \frac{98}{3}$

Example 2: Find the area that is bounded by $y = 5 - x^2$ and $y = 2 - 2x$.

$f(x) = 5 - x^2 \quad g(x) = 2 - 2x$

① Find intersection of two curves

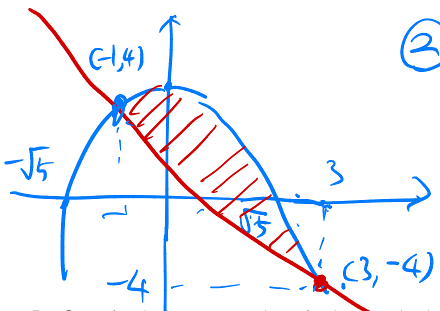
$f(x) = g(x), \quad \boxed{5 - x^2} = 2 - 2x.$

$\Rightarrow \quad x^2 - 2x + 2 - 5 = 0$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$\Rightarrow$ Two intersection pts $(-1, 4), (3, -4)$



② Figure out whether $f \geq g$ or $g \geq f$

$0 \in [-1, 3]$

$\Rightarrow f(0) = 5, \quad g(0) = 2.$

$\boxed{f \geq g}$

③ Calculate area

$\int_{-1}^{3} f(x) - g(x)\, dx$

$= \int_{-1}^{3} (5 - x^2 - 2 + 2x)\, dx$

$= \int_{-1}^{3} (-x^2 + 2x + 3)\, dx$

$= \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^{3}$

$= -\frac{1}{3}3^3 + 9 + 9$

$\quad - (+\frac{1}{3} + 1 - 3)$

$= -9 + 9 + 9$

$\quad - (-\frac{5}{3})$

$= 9 + \frac{5}{3} = \boxed{\frac{32}{3}}$

$\approx 10.6667.$

Example 3: Find the area that is bounded by $y = \ln x$ and $y = 1$ on $[1, 5]$.

① Intersection

$f(x) = \ln x$

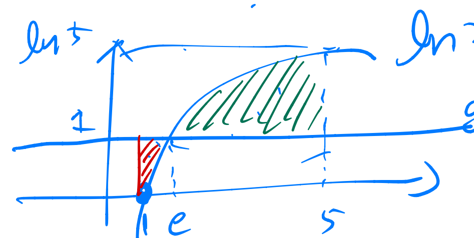
$g(x) = 1$

$f(1) = \ln 1 = 0$

$f(5) = \ln 5.$

$\ln x = 1$

$\Rightarrow x = e \approx 2.71883 \ldots$



$\ln x = f(x)$

② $[1, e], [e, 5]$

If $x \in [1, e]$,

$f(x) < 1 \Rightarrow g(x) \geq f(x)$

If $x \in [e, 5]$

$f(x) > 1 \Rightarrow g(x) \leq f(x)$

fnInt(1 - ln x, x, 1, e)

+fnInt(ln x - 1, x, e, 5)

$\downarrow$

$1.4838 \approx$

Red + Green

$= 2(e - \frac{1}{e}) - 5 + \frac{1}{5}.$

③

Red Area: $\int_1^e g(x) - f(x)\, dx$

$= \int_1^e (1 - \ln x)\, dx$

$= (e - 1) - (\frac{1}{e} - \frac{1}{1})$

$= e - 1 - \frac{1}{e} + 1$

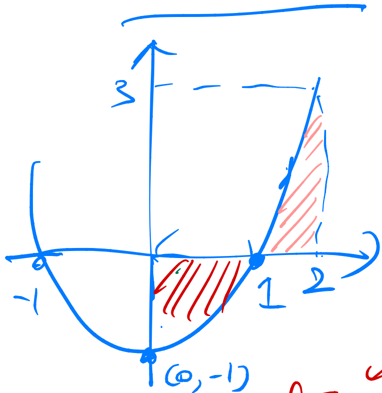
$= e - \frac{1}{e}.$

Green Area $:= \int_e^5 f(x) - g(x)\, dx$

$= \int_e^5 (\ln x - 1)\, dx \quad = \left[\frac{1}{x} - x\right]_e^5$

$= \frac{1}{5} - 5 - \frac{1}{e} + e$

Example 4: Find the area that is bounded by $y = x^2 - 1$ and the x-axis on $[0, 2]$.



$f(x) = x^2 - 1$ $\quad g(x) = 0$

$f(2) = 2^2 - 1$
$= 3$

Red Area $= \int_0^1 -f(x)\,dx$

fnInt$(-x^2+1, x, 0, 1)$ $= \int_0^1 (-x^2+1)\,dx$
$= .6667$

$= -\int_0^1 x^2\,dx + \int_0^1 1\,dx$

$= -\left(\frac{1}{3}x^3 \Big|_0^1\right) + 1 - 0$

$= -\frac{1}{3} + 1 = \frac{2}{3}$

Red + Pink

$= \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$

fnInt$(x^2-1, x, 1, 2)$ ← Pink Area $= \int_1^2 f(x) = \int_1^2 x^2 - 1\,dx$
$= 1.3334$

$= \left(\frac{1}{3}x^3 - x\right)\Big|_1^2 = \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)$

$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

Example 5: Find the area that is bounded by $y = -x^2$ and $y = 2x^3 - 5x$.

$f(x) = -x^2$

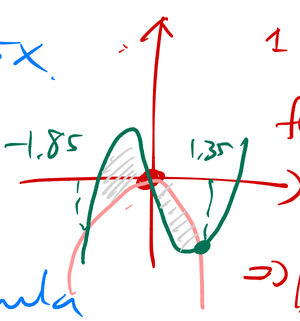
$-x^2 = 2x^3 - 5x$ $\qquad g(x) = 2x^3 - 5x$

$\Rightarrow 0 = 2x^3 + x^2 - 5x$
$= x(2x^2 + x - 5)$
$= x$

quadratic formula to figure out intersection

Calculator to figure out intersection

$x = 0,\ 1.3508,\ -1.8508$



$1 \in [0, 1.3508)$
$f(1) = -1$
$g(1) = 2 - 5$
$= -3$
$\Rightarrow \boxed{f \geq g}$ on $[0, 1.3508]$

$-1 \in [-1.8508, 0]$
$f(-1) = -1$
$g(-1) = -2 + 5 = 3$
$g \geq f$ on $[-1.8508, 0]$

③ $\int_{-1.8508}^{0} g(x) - f(x)\,dx = \int_{-1.8508}^{0} (2x^3 - 5x + x^2)\,dx$

$+ \int_0^{1.3508} f(x) - g(x)\,dx = \int_0^{1.3508} (-x^2 - 2x^3 + 5x)\,dx$ $\quad \boxed{6.8854}$

$f(x) = x^2 - x \quad g(x) = 2x$

Example 6: Find the area that is bounded by $y = x^2 - x$ and $y = 2x$ on $\boxed{-2 \le x \le 4}$.

$[-2, 4]$

① Intersection

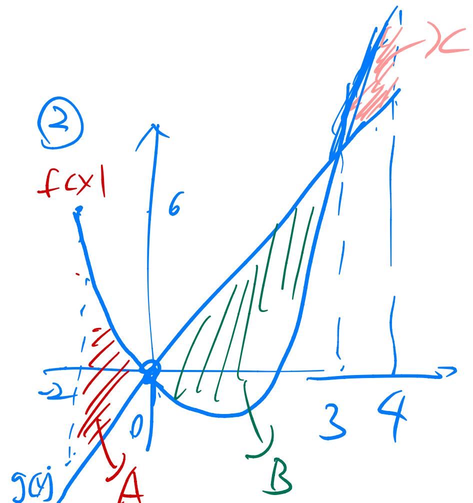
$f(x) = g(x) \Rightarrow x^2 - x = 2x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$\Rightarrow x = 0$ or $3 \Rightarrow [-2, 0], [0, 3], [3, 4]$

②



$A = \int_{-2}^{0} f(x) - g(x)\, dx$

$= \int_{-2}^{0} (x^2 - 3x)\, dx$

$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_{-2}^{0} = -\left(\frac{1}{3}(-8) - \frac{3}{2}\cdot 4\right)$

$= \frac{8}{3} + 3 = \boxed{\frac{17}{3}}$

$C = \int_3^4 (x^2 - 3x)\, dx$

$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_3^4$

$= \boxed{\frac{46}{3} + \frac{9}{2}} =$

$21 + \frac{1}{3}$

$15 + \frac{1}{3}$

$\left(\frac{64}{3} - \frac{12}{1}\right)$

$-\left(9 - \frac{27}{2}\right)$

$+9$

$B = \int_0^3 (g(x) - f(x))\, dx$

$= \int_0^3 (2x - x^2 + x)\, dx = \int_0^3 (-x^2 + 3x)\, dx = \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2\right]_0^3$

$= -\frac{1}{3}\cdot 27 + \frac{27}{2} = -9 + \frac{27}{2}$

$\boxed{= \frac{9}{2}}$

$\frac{46}{3} + \frac{9}{2} + \frac{17}{3} + \frac{9}{2}$

$= 21 + 9 = \boxed{30}$

Example 7: Set up the definite integral(s) representing the area bounded by $y = -x^2 + 10x - 17$ and the x-axis on $[5, B]$, where $B > 8$.

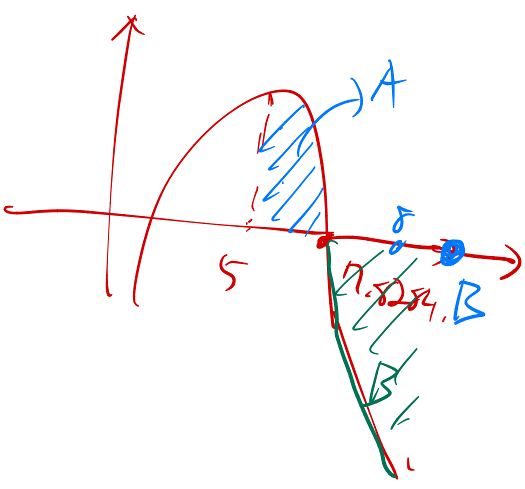
$g(x) = 0 \quad f(x) = -x^2 + 10x - 17$

quadratic formula

(calculator)

$x = \boxed{7.8284} < 8$

(2.1716)



$A = \int_5^{7.8284} f(x)\, dx = \boxed{\int_5^{7.8284} (-x^2 + 10x - 17)\, dx}$

$B = \int_{7.8284}^{B} -f(x)\, dx = \boxed{\int_{7.8284}^{B} (+x^2 - 10x + 17)\, dx}$

$= \left[\frac{1}{3}x^3 - 5x^2 + 17x\right]_{7.8284}^{B}$

$= \frac{1}{3}B^3 - 5B^2 + 17B - \left(\frac{1}{3}(7.82)^3 - 5\cdot(7.82)^2 + 17x\right)$