Feel free to ask Question!

+ Please bring your Calculator!

Section 6.5: The Fundamental Theorem of Calculus

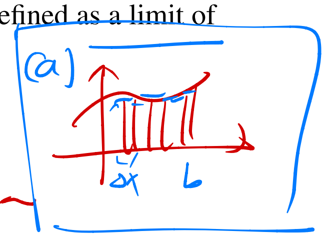
Recall from Section 6.4: Upload your total scores for 100 scores.

- If f is a continuous function on $[a, b]$, then the definite integral of f from a to b can be defined as a limit of a Riemann sum for the function f:

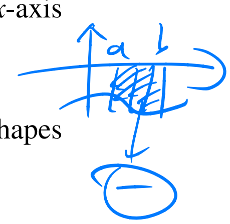
$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

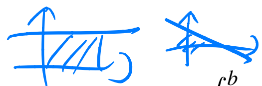
height of rectangle — width



- $\int_a^b f(x)\,dx$ gives an exact value and "counts" area **above** the x-axis **positively** and area **below** the x-axis **negatively** .



- We can attempt to use the graph of $f(x)$ to interpret $\int_a^b f(x)\,dx$ in terms of areas (i.e., use geometric shapes between $f(x)$ and the x-axis to find an exact answer when possible).



- If it is not possible to use geometric shapes to find an exact value, we can **estimate** $\int_a^b f(x)\,dx$ by using a Riemann sum.

① Right Sum ② Left Sum ③ Midpoint Sum

One of the things we will learn in this section is how to evaluate a definite integral exactly (without looking at the graph of the function/using geometric shapes).

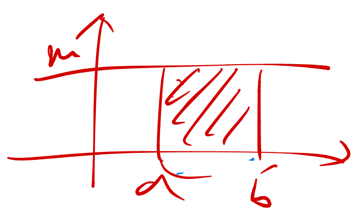
$\int$ indefinite

$\int_a^b$ definite

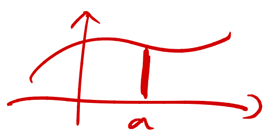
Properties of the Definite Integral- The following are properties for continuous functions f and g.

1. $\boxed{\int_a^b m\,dx} = m(b-a)$, where m is a constant

height — width



2. $\int_a^a f(x)\,dx = 0$



3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

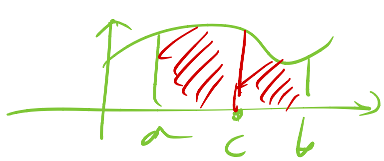
$a-b = -(b-a)$

indefinite also

4. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant

5. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$



Example 1: If it is known that $\int_1^4 f(x)\,dx = 7.5$, $\int_1^4 g(x)\,dx = 21$, and $\int_4^5 g(x)\,dx = 61/3$, find

a) $\int_1^4 (4g(x) - 9f(x))\,dx$

$\overset{(5)}{=} \int_1^4 4g(x)dx - \int_1^4 9f(x)\,dx$

$\overset{(4)}{=} 4\int_1^4 g(x)dx - 9\cdot\int_1^4 f(x)\,dx$

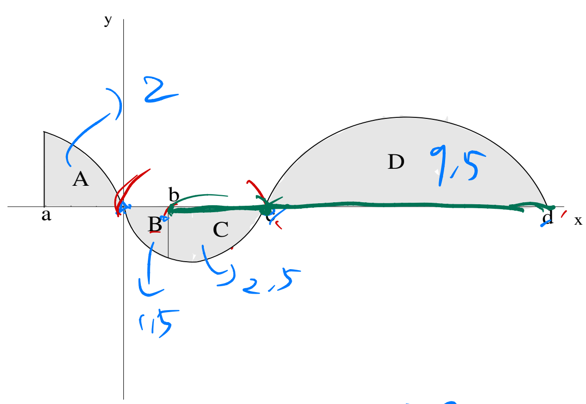
$= 4\cdot 21 - 9\cdot 7.5 = 16.5$

b) $\int_1^5 (-4g(x))\,dx$

$\overset{(6)}{=} \int_1^4 (-4g(x))dx + \int_4^5 (-4g(x))dx$

$\overset{(4)}{=} -4\cdot\int_1^4 g(x)dx - 4\cdot\int_4^5 g(x)dx = -4\cdot 21 - 4\cdot\frac{61}{3} = \frac{-496}{3}$

Example 2: Use the graph of $f(x)$ with the indicated areas below to answer the following.



area of A:	2.0
area of B:	1.5
area of C:	2.5
area of D:	9.5

area A $= \int_a^0 f(x)\,dx$

area B $= -\int_0^b f(x)\,dx$

area C $= -\int_b^c f(x)\,dx$

area D $= \int_c^d f(x)\,dx$

a) Find $\int_a^c f(x)\,dx - \int_0^c 2f(x)\,dx$.

$\overset{(6)}{=} \underbrace{\int_a^0 f(x)dx}_{\text{area } A} + \underbrace{\int_0^c f(x)\,dx}_{-(\text{area } B + C)} - \int_0^c 2f(x)dx$

$\overset{(4)}{=} \quad - 2\cdot\int_0^c f(x)\,dx$

$\overset{(6)}{=} \quad -2\left(\underbrace{\int_0^b f(x)\,dx}_{-\text{area } B} + \underbrace{\int_b^c f(x)dx}_{-\text{area } C}\right)$

$= A - (B+C) - 2(-B-C)$

$= A - \cancel{B} - \cancel{C} + \cancel{2}B + \cancel{2}C$

$= A + B + C \quad = 2.0 + 1.5 + 2.5 =$

b) Find $\int_0^0 4f(x)\,dx + \int_d^b f(x)\,dx$.

$\overset{(6)}{}$ $d > b$

$= -\int_b^d f(x)dx$

$= -\left(\int_b^c f(x)dx + \int_c^d f(x)dx\right)$

$= -(-C + D)$

$= C - D = 2.5 - 9.5 = -7$

We can evaluate a definite integral exactly using the Fundamental Theorem of Calculus, Part 2:

The Fundamental Theorem of Calculus, Part 2 - Suppose f is continuous on $[a,b]$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

indefinite integral.

Example 3: Evaluate the following integrals:

a) $\int_1^6 (e^x + x)\,dx. = \int_1^6 e^x\,dx + \int_1^6 x\,dx$

(① indefinite integral first)

$\left(\int e^x dx = e^x + c, \quad \int x^1 dx = \frac{1}{1+1} x^{1+1} = \frac{1}{2}x^2 + c \right)$ $c=0$, $c=0$

② FTC $= (e^6 - e^1) + \left(\frac{1}{2}(6)^2 - \frac{1}{2}(1)^2\right)$

$= e^6 - e + \frac{1}{2}(36-1) = e^6 - e - \frac{35}{2}$

(1) b) $\int_0^2 \sqrt{4x+1}\,dx.$

u-substitution

$u = 4x+1$, $du = 4\,dx$

$u(2) =$

$u(0) =$

$u = 4x+1$, $u = 4\cdot 2+1 = 9$, $\int \sqrt{4x+1}\,dx$, $u = 4\cdot 0+1 = 1$

(2) $\frac{1}{4}\int_{0=x}^{2=x} \sqrt{u}\,\frac{1}{4}\,du$

$= \frac{1}{4}\left(\frac{1}{\frac{1}{2}+1} u^{1+\frac{1}{2}} \Big|_{0=x}^{2=x}\right)$

$= \frac{1}{4}\left(\frac{1}{\frac{1}{2}+1}(4x+1)^{1+\frac{1}{2}} \Big|_0^2\right)$

$= \frac{1}{4}\cdot\frac{2}{3}\cdot\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$

$= \int_1^9 \sqrt{u}\,\frac{1}{4}du$

$\frac{1}{4}du = dx$

(4) $= \frac{1}{4}\int_1^9 \sqrt{u}\,du = \frac{1}{4}\cdot\left(\frac{1}{\frac{1}{2}+1}\cdot u^{1+\frac{1}{2}} \Big|_1^9\right)$

$u^{\frac{1}{2}}$

$= \frac{1}{4}\cdot\frac{2}{3}\cdot\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{1}{2}\cdot\frac{1}{3}(27-1) = \boxed{\frac{13}{3}}$

Using $fnInt$ on the Calculator - We can estimate the definite integral $\int_a^b f(x)\,dx$ using the calculator:

1. Type $f(x)$ into Y_1 by pressing the $y =$ button.

2. From your homescreen, press MATH and then choose option 9:fnInt(

fnInt(integrand (function), variable x, a, b)

region $[a,b]$

Example 4: Find $\int_1^6 (e^x + x)\,dx$ again using $fnInt$ (from Example 3). Round your answer to 4 decimal places.

$= 418.210517 = 418.2105$

Ti-83

fnInt(□, □, □, □)

fnInt($e^x + x$, x, 1, 6)

Example 5: Evaluate $\int_2^K (t^2+4)\,dt$ $= \int_2^K t^2\,dt + \int_2^K 4\,dt$

$\overset{FTC}{=} \frac{1}{3}t^3 \Big|_2^K + 4\cdot(K-2)$

$= \frac{1}{3}K^3 - \frac{1}{3}2^3 + 4K - 8$

$K=10$ $\quad = \frac{1}{3}K^3 - \frac{8}{3} + 4K - 8 = \frac{1}{3}K^3 + 4K - \frac{32}{3}$

Interpreting the Definite Integral as Change - We can also interpret the definite integral of a rate of change function as the **change** in its antiderivative from $x=a$ to $x=b$:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

FTC

$\int f'(x) = f(x) + C$

Example 6: A honeybee population starts with 200 honeybees and increases at a rate of $n'(t) = 100e^{2t}$ bees per week, where t is in weeks and $t \geq 0$.

a) Find the change in the honeybee population over the first 4 weeks. Round to the nearest integer, if necessary.

fnInt$(100e^{2x}, x, 0, 4)$

$n(4) - n(0) = \int_0^4 n'(t)\,dt = \int_0^4 100\cdot e^{2t}\,dt = 100\int_0^4 e^{2t}\,dt$

(1) Use calculator to calculate definite integral.

$= 148997.8994.$

≈ 148998 bees

(2) FTC

$u = 2t$

$du = 2\,dt$

$\frac{1}{2}du = dt$

$u(0) = 0$, $u(4) = 8$

$= 100\int_0^8 e^u \left(\frac{1}{2}\right)du$

$= \frac{100}{2}\int_0^8 e^u\,du$

$= 50\cdot(e^8 - 1)$

b) Find the change in the honeybee population over the second 4 week period. Round to the nearest integer, if necessary.

$\int_4^8 n'(t)\,dt \approx 444,156,478$ bees

calculator

c) What will be the total change in the honeybee population during the seventh and eighth weeks? Round to the nearest integer, if necessary.

$\int_6^8 n'(t)\,dt \approx 436,167,787$ bees.

calculator