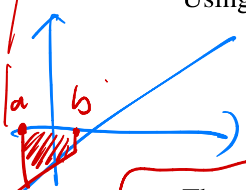
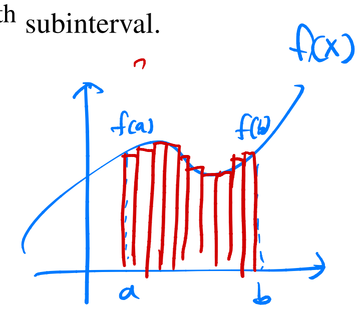
Feel free to ask questions!

Section 6.4: The Definite Integral

Recall from Section 6.3:

- For a continuous function $f(x)$, where $f(x) \geq 0$, we can estimate the area of a region that lies under $f(x)$ from $x = a$ to $x = b$ by dividing the region into subintervals (rectangles) and adding the areas of the rectangles.
- In general, we can use any x-coordinate, x_i^*, to find the the height of the rectangle in the i^{th} subinterval.





Using summation notation, we can write the sum of the areas of the rectangles as

height of 1st rectangula

$$\underbrace{f(x_1^*)}_{}\underbrace{\Delta x}_{\text{width}}+\underbrace{f(x_2^*)}_{\text{height of 2nd rectangula}}\Delta x+\cdots+f(x_n^*)\Delta x=\boxed{\sum_{i=1}^{n} f(x_i^*)\Delta x} \approx$$

Area ≈ sum of rectangulars.
① Left sum ② Right sum ③ Midpt sum

- The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann sum.

$f(x)<0$ ⇒ area will be given with (−) sign.

- We can estimate the distance an object travels by estimating the area under its velocity curve using a Riemann sum (assuming the velocity function is greater than or equal to zero).

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

The Definite Integral

$\boxed{f(x_i^*)<0}$

If we let the number of subintervals (n) go to infinity, then we get the actual or exact area of the region under $f(x)$ between $x = a$ and $x = b$, assuming $\boxed{f(x) \geq 0}$. In other words:

Theorem: $\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \text{Area between } f(x) \text{ and } x\text{-axis}}$

The above limit occurs so much, that it is given a special name and notation. We refer to this common limit as the **definite integral** of $f(x)$ from a to b and write it as

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\underbrace{\Delta x}_{\frac{b-a}{n}}}$$

Definition of a Definite Integral: Given a function $f(x)$ that is continuous on the interval $[a,b]$, we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then, the **definite integral of $f(x)$ from a to b** is

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

NOTE: $\int_a^b f(x)\,dx$ "counts" area above the x-axis as positive and area below the x-axis as negative. Thus, if $f(x) \geq 0$, the definite integral represents the actual area, and if $f(x) < 0$, we say it represents the *signed* area. If the function is both positive and negative, then we say the definite integral represents the *accumulated or net* area.

* $\int f(x)dx$ = function with constant

$\int_a^b f(x)dx$ = number (maybe negative)

Important Notes:

1. In the notation $\int_a^b f(x)dx$, the symbol $\int$ is called an **integral sign**. It is an elongated S (since it is a limit of sums). $f(x)$ is called the **integrand** and a and b are the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The symbol dx has no official meaning by itself; $\boxed{\int_a^b f(x)dx}$ is all one symbol. The procedure of calculating an integral is called **integration.**

   = # number.

2. The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on x. Recall that an indefinite integral, $\int f(x)\,dx$, represents a family functions.

Example 1: Use the graph of $f(x)$ below to find the following. Note that the graph consists of three straight lines and a semicircle.

In $x \in (-5,2)$, $f(x)<0$

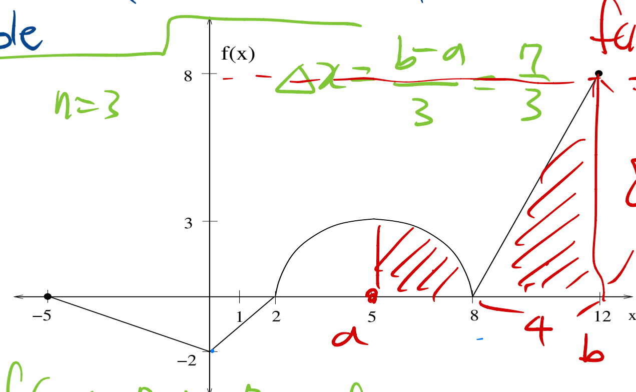
a) $\int_{-5}^{2} f(x)\,dx$

$= -\left((b-a)\cdot \text{height of triangle}\right) = -(2-(-5))\cdot 2 = -14$

$f(x) = \begin{cases} -\frac{2}{5}x - 2 & -5 < x < 0 \\ x - 2 & 0 \le x < 2 \end{cases}$

$n=3$

$\Delta x = \frac{b-a}{3} = \frac{7}{3}$



$f(-5+\frac{7}{3})\cdot\frac{7}{3} + f(-5+\frac{14}{3})\cdot\frac{7}{3}$

$\approx -14.$

b) $\boxed{\int_{0}^{8} f(x)\,dx}$ = Area of semicircle − Area of triangle

$= \pi\cdot 3^2\cdot\frac{1}{2} - 2\cdot 2\cdot\frac{1}{2}$

$= \frac{9}{2}\pi - 2$

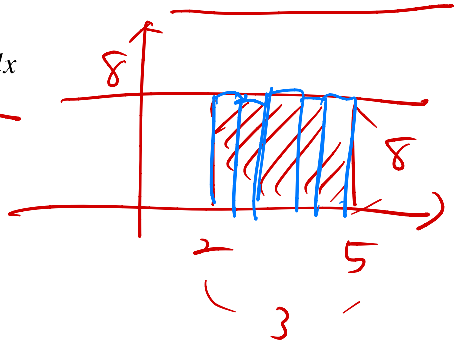
c) $\boxed{\int_{5}^{12} f(x)\,dx}$ = Area of quarter of the circle + Area of triangle.

$= \pi\cdot 3^2\cdot\frac{1}{4} + 4\cdot 8\cdot\frac{1}{2}$

$= \frac{9}{4}\pi + 16$

Example 2: Evaluate each of the following by interpreting the definite integral in terms of areas.

a) $\int_2^5 8\,dx$



$3 \cdot 8 = 24$.

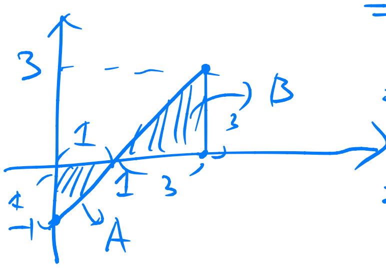
$\Rightarrow L_n = R_n = M_n = \int_2^5 8\,dx = 24$

b) $\int_0^4 (x-1)\,dx$ $= -A + B$

$= -1 \cdot 1 \cdot \frac{1}{2} + 3 \cdot 3 \cdot \frac{1}{2}$

$= -\frac{1}{2} + \frac{9}{2}$

$= \frac{8}{2} = 4$.



$\Delta x = \frac{3-0}{n}$

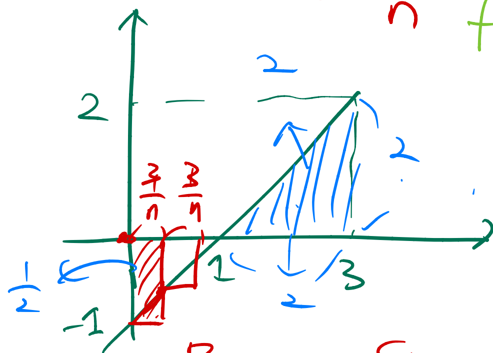
$f(x_0^*) = a = 0$ Height of first rect.

$x_1^* = 0 + \frac{3}{n} = \frac{3}{n}$

$x_2^* = 0 + \frac{6}{n} = \frac{6}{n}$

$x_k^* = 0 + \frac{3k}{n} = \frac{3k}{n}$

$x_n^* = b = 3$



Riemann Sum

$\sum_{i=0}^{n-1} f(x_i^*)\Delta x = \sum_{i=0}^{n-1} \left(\frac{3i}{n} - 1\right)\frac{3}{n}$

Question: What do we do if we cannot use geometric shapes between $f(x)$ and the x-axis to find $\int_a^b f(x)\,dx$ exactly?

$= \sum_{i=0}^{n-1} \left(\frac{9i}{n^2} - \frac{3}{n}\right)$

$= \frac{9}{n^2}\sum_{i=0}^{n-1} i - 3$

$= \frac{9}{n^2}\left(\frac{n(n-1)}{2}\right) - 3$

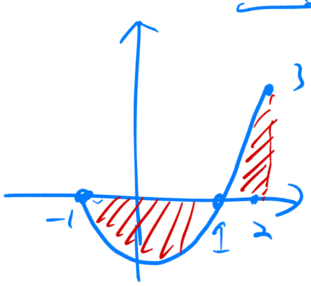
$= \frac{9n^2 - 9n}{2n^2} - 3$

$\int_0^3 f(x)\,dx = \lim_{n\to\infty}\left(\frac{9n^2-9n}{2n^2} - 3\right)$

$= \frac{9}{2} - 3 = \frac{3}{2}$

Area of B $-$ Area of A $= 2 - \frac{1}{2} = \frac{3}{2}$

Example 3: Use a midpoint sum with $n = 3$ to estimate $\int_{-1}^2 (x^2 - 1)\,dx$.



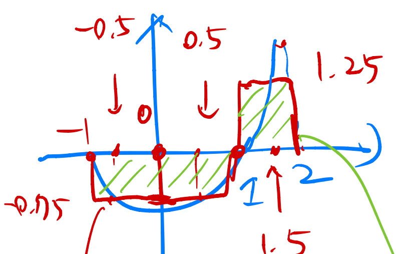
$f(x) = x^2 - 1$

$f(-1) = (-1)^2 - 1 = 1 - 1 = 0$

$f(2) = 2^2 - 1 = 3$

$f(0) = -1$

$\Delta x = \frac{2-(-1)}{3} = \frac{3}{3} = 1$



$f(0.5) = f(-0.5) = 0.25$
-1
$= -0.75$

$f(1.5) = 2.25 - 1 = 1.25$

Note: In Section 6.5, we will learn how to evaluate a definite integral exactly without using a graph/geometric shapes!

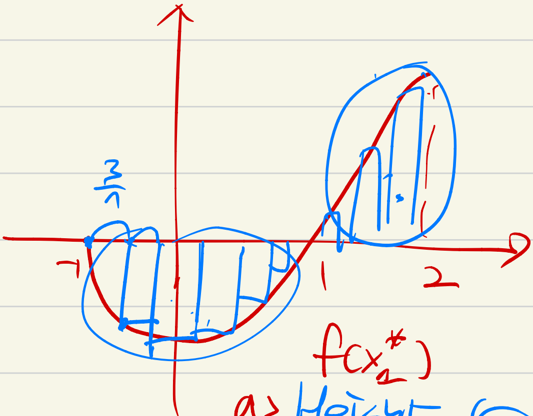
$-2 \times 0.75 + 1 \cdot 1.25 = -1.5 + 1.25$

$= \boxed{-0.25 \approx \int_{-1}^2 (x^2-1)\,dx}$

$$\int_{-1}^{2} (x^2-1)\, dx$$

$f(x) = x^2 - 1$

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$



Right sum

$f(x_1^*)$ as Height of first rectangular

$x_0 = a = -1$

$x_1 = a + \Delta x = -1 + \frac{3}{n} = \frac{-n+3}{n}$

$x_2 = a + 2\cdot\Delta x = -1 + \frac{6}{n} = \frac{-n+6}{n}$

$\vdots$

$x_k = a + k\cdot\Delta x = -1 + \frac{3\cdot k}{n} = \frac{-n+3k}{n}$

$x_n = b = 2 \left(= \frac{-n+3n}{n} = \frac{2n}{n} \right)$

Rieman Sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \left(\left(\frac{-n+3i}{n}\right)^2 - 1 \right) \cdot \frac{3}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{n^2 - 6ni + 9i^2}{n^2} \cancel{-1} \right) \cdot \frac{3}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{-6ni + 9i^2}{n^2} \right) \frac{3}{n} = \sum_{i=1}^{n} \frac{-18ni + 27i^2}{n^3}$$

$$= \sum_{i=1}^{n} \left(-\frac{18}{n^2}\cdot i + \frac{27}{n^3}\cdot i^2 \right) = \frac{-18}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \cdot \sum_{i=1}^{n} i^2$$

$$= \frac{-18}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \cdot \sum_{i=1}^{n} i^2$$

$$\left(* \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \frac{-18}{n^2} \cdot \frac{n^2+n}{2} + \frac{\cancel{27}^{\,9}}{n^3} \frac{n(n+1)(2n+1)}{\cancel{6} \cdot 2}$$

$$= -9 \cdot \left(\frac{n^2+n}{n^2} \right) + \frac{9}{2} \frac{n(n+1)(2n+1)}{n^3}$$

$$R_n = -9 \left(\frac{n^2+n}{n^2} \right) + \frac{9}{2} \left(\frac{2n^3 + \sim}{n^3} \right)$$

$\rightarrow 1$ when $n \to \infty$ $\qquad \rightarrow 2$.

$$\lim_{n \to \infty} R_n = -9 \cdot 1 + \frac{9}{2} \cdot 2 = 0$$

$$\int_{-1}^{2} f(x)\,dx = 0.$$