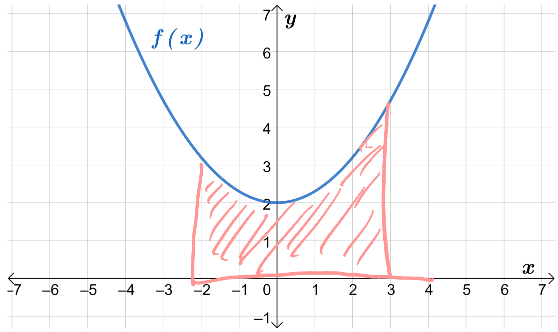
Feel free to question me!

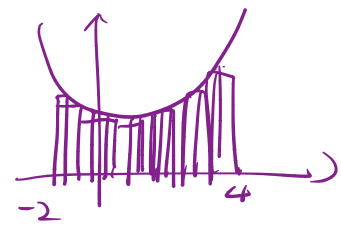
Section 6.3: Estimating Distance Traveled

Definite Integral

The Area Problem

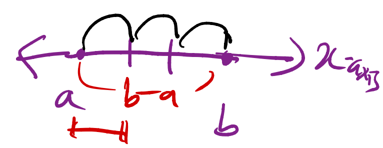
Consider the area of a region that lies under the curve of of $f(x) = 0.3x^2 + 2$ from $x = -2$ to $x = 4$:





We will estimate the area under a curve from $x = a$ to $x = b$ by dividing the region into n subintervals (rectangles) of equal width.

$$\text{width of each subinterval} = \boxed{\Delta x} = \frac{b-a}{n}$$



where n is the number of subintervals (rectangles).

① Left Sum ② Right Sum ③ Midpoint Sum

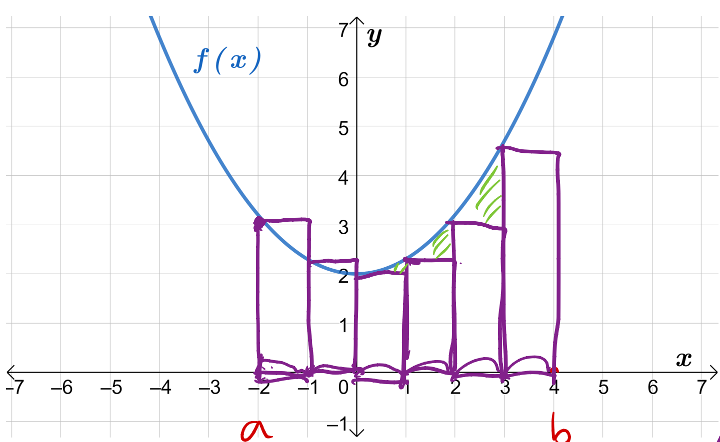
$\Delta x = \frac{b-a}{3}$ $n=3$

Left Sum

To estimate the area under a curve using a **left-hand sum**, we calculate the height of each rectangle by evaluating the function at the left endpoint of the subinterval.

Estimating the area of such a region using left endpoints of n subintervals is denoted L_n.

Example 1: For the function $f(x) = 0.3x^2 + 2$, estimate the area of the region that lies under the graph of $f(x)$ between $x = -2$ to $x = 4$ using a left-hand sum with six subintervals of equal width.



$n = 6$

$b = 4$

$a = -2$

$\Delta x = \frac{4-(-2)}{6} = \frac{6}{6} = 1$

$L_n = f(-2)\cdot 1 + f(-1)\cdot 1 + f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 = 17.7$

Sum of all rectangulars

⇒ Underestimate

Right Sum

② Right Sum

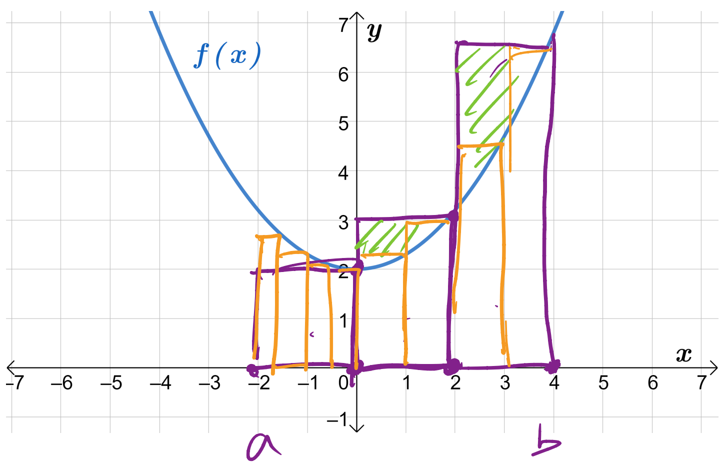
To estimate the area under a curve using a **right-hand sum**, we calculate the height of each rectangle by evaluating the function at the right endpoint of the subinterval.

Estimating the area of such a region using right endpoints of n subintervals is denoted R_n.

Example 2: For the function $f(x) = 0.3x^2 + 2$, estimate the area of the region that lies under the graph of $f(x)$ between $x = -2$ to $x = 4$ using a right-hand sum with three subintervals of equal width.

$n=3$

$$\Delta x = \frac{b-a}{n} = \frac{4-(-2)}{3} = \frac{6}{3} = 2$$



$$\boxed{R_3 = f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2}$$

green side
⇒ overestimates

⇒ $\boxed{\text{Left Sum} \le \text{Actual area} \le \text{Right sum}}$

Notice: For the above function, L_6 is an underestimate of the actual area, and R_3 is an overestimate. Be careful! You have to graph the function with the rectangles to determine if a sum is an overestimate or underestimate of the actual area.

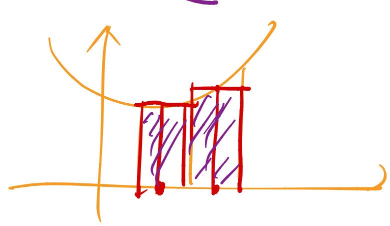
How can we obtain a better estimate? increasing n.

⇒ Taking width smaller and smaller to get the better estimates.

General Sum

We can use *any* x-coordinate in each subinterval to get the height of the rectangle (not just the left or right endpoints). Another sum that is often used is called a **midpoint sum**. In this type of sum, the x-coordinates that are used to find the height of each rectangle are the midpoints of the subintervals.

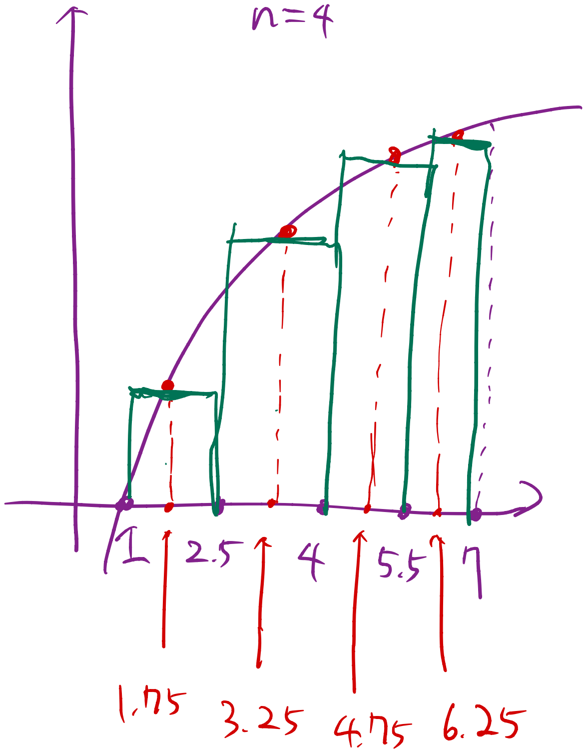
Estimating the area using midpoints of n subintervals is denoted M_n.



$a=1 \quad b=7$

Example 3: Estimate the area of the region that lies under the graph of $f(x) = 4\ln x$ between $x = 1$ and $x = 7$ using a midpoint sum with four subintervals of equal width.

$n=4$



$$\Delta x = \frac{7-1}{4} = \frac{6}{4} = 1.5$$

$b=7$

$a=1$

$n=4$

$$M_4 = f(1.75)\cdot 1.5 + f(3.25)\cdot 1.5 + f(4.75)\cdot 1.5 + f(6.25)\cdot 1.5 \approx 30.7740$$

1st subinterval $(1, 2.5)$ $x_1^* = 1.75$

2nd ~ $(2.5, 4)$ $x_2^* = 3.25$

3rd ~ $(4, 5.5)$ $x_3^* = 4.75$

4th ~ $(5.5, 7)$ $x_4^* = 6.25$

Notation

Recall in the previous example (Example 3), we calculated the following midpoint sum to estimate the area under $f(x) = 4\ln x$ between $x = 1$ and $x = 7$ using four subintervals:

$$M_4 = f(1.75)\cdot(1.5) + f(3.25)\cdot(1.5) + f(4.75)\cdot(1.5) + f(6.25)\cdot(1.5)$$

If we let x_i^* be the x-coordinate we use to get the height of the rectangle in the i^{th} subinterval, and let Δx be the width of each subinterval, we can write the above sum as

$$M_4 = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

$n=4$

$\sum_{i=1}^{n} i = 1+2+3+\cdots+n$

$\sum_{i=1}^{n} x_i = x_1+x_2+x_3+\cdots+x_n$

Ex) If $x_1=2$ $x_2=3$ $x_3=7$ $x_4=2$

$\sum_{i=1}^{4} x_i = 2+3+7+2 = 14$

$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x$

constant

In general, we can write any sum with n subintervals as

$$M_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

Now, we can write this sum more easily using **sigma** notation:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

when $n \to \infty$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

NOTE: The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum.**

a, b : original interval

$dx \sim \Delta x$

$V = 4$ km/h.

Application: The Distance Problem

C.S → H [4km/h, 2hour] ⇒ $2 \cdot 4 = 8$km

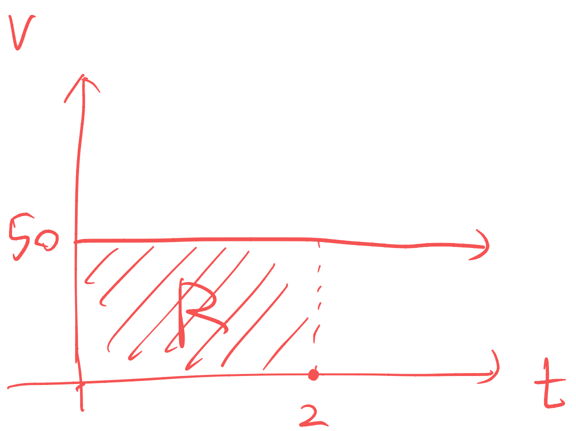
One common application of estimating the area under a curve using a Riemann sum is estimating the distance an object has traveled (hence the title of this section!).

If an object travels at a constant rate (i.e., velocity), we can determine how far it has traveled after a certain amount of time (recall that distance = rate × time).

Example 4: Suppose a car travels at a $\boxed{\text{constant rate}}$ of $\boxed{\text{50 miles per hour}}$ for 2 hours. What is the total distanced traveled?

$V = 50$ m/h $h = 2$

$V \cdot h = 50 \cdot 2 = 100$ m



Area of R

$= V \cdot h$

Note: From the previous example, we see that the $\boxed{\text{distance an object travels}}$ is equal to $\boxed{\text{the area under the velocity curve}}$ (assuming the velocity function is positive).

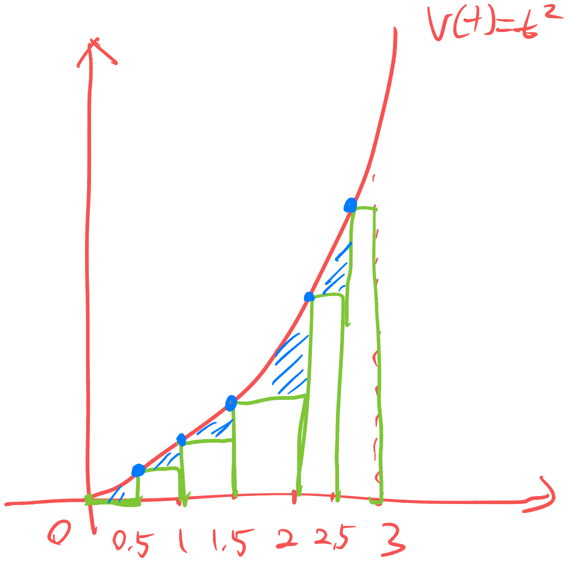
However, unlike the car in the previous example, velocity is usually not constant. In other words, it will vary over time. We can estimate the distance traveled by using a Riemann sum to estimate the area under the velocity curve.

Example 5: An object travels with velocity $\boxed{v(t) = t^2}$, where v is in feet per second and t is in seconds. Estimate how far the object traveled during the first $\boxed{\text{three seconds}}$ using $\boxed{\text{left endpoints}}$ and $\boxed{\text{six subintervals}}$ of $\boxed{\text{equal width.}}$

$a \sim t=0$ $\quad b = 3 \sim b$ $\quad n = 6$

$V(t) = t^2$

$$\Delta x = \frac{3-0}{6} = \frac{1}{2}$$



$$L_6 = V(0)\cdot 0.5 + V(0.5)\cdot 0.5 + V(1)\cdot 0.5 + V(1.5)\cdot 0.5 + V(2)\cdot 0.5 + V(2.5)\cdot 0.5$$

$$= \cancel{0 \cdot 0.5} + 0.25 \cdot 0.5 + 1 \cdot 0.5 + 2.25 \cdot 0.5 + 4 \cdot 0.5 + 6.25 \cdot 0.5$$

$$= 6.875 \text{ ft}$$

It underestimates the blue region.

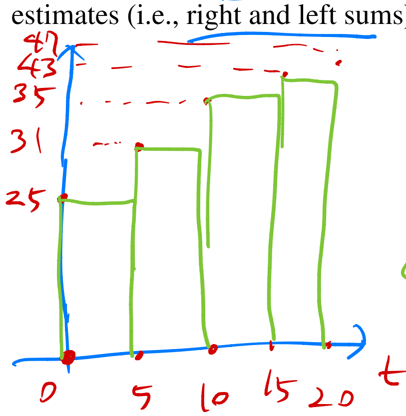
$\Delta x = \frac{20-0}{4} = 5$

We can also estimate distance traveled if the velocity of an object is recorded at certain times and displayed in a table.

$b=20$, $a=0$, $n=4$, $\Delta x = 5$ sec.

Example 6: The table below shows the velocity (ft/s) of an object every five seconds over a 20 second time interval. Estimate the total distance the object travels over the 20 second time interval by finding upper and lower estimates (i.e., right and left sums).

Time (s)	0	5	10	15	20
Velocity (ft/s)	25	31	35	43	47

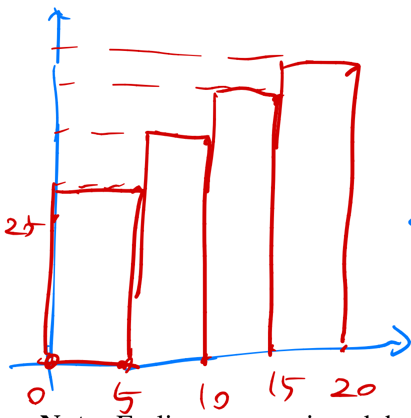


Left sum

$L_4 = v(0)\cdot 5 + v(5)\cdot 5 + v(10)\cdot 5 + v(15)\cdot 5$

$= 25\cdot 5 + 31\cdot 5 + 35\cdot 5 + 43\cdot 5$

$= \boxed{670 \text{ ft.}}$ Left sum: Lower sum



Right sum

$R_4 = v(5)\cdot 5 + v(10)\cdot 5 + v(15)\cdot 5 + v(20)\cdot 5$

$= 31\cdot 5 + 35\cdot 5 + 43\cdot 5 + 47\cdot 5 = \boxed{780 \text{ ft}}$ Right sum Upper sum

Note: Earlier, we mentioned that increasing the number of subintervals (rectangles) would give us a better estimate of the area under a curve. In fact, if we let the *number of rectangles go to infinity* in our Riemann sum, we will get an exact answer! We will explore this idea in the next section (Section 6.4).

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x} = \boxed{\text{exact area of given region}}$$