## Section 6.2: Substitution

$u = 4x^2 + 7$

$\frac{du}{dx} = 8x$

### Reversing the Chain Rule

Recall that in order to find the derivative of a composite function, we can use the Chain Rule:

$8u^7 \cdot$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$\frac{d}{dx}(\underbrace{4x^2+7}_{u})^8 = 8(4x^2+7)^7(8x)$ chain rule

In this section, we will look at reversing the Chain Rule in order to calculate an indefinite integral in which the integrand involves a Chain Rule. In other words,

$= 64x(4x^2+7)^7$

$$\int \boxed{f'(g(x))} \cdot \boxed{g'(x)} dx = f(g(x)) + C$$

$\int 64x(4x^2+7)^7 dx$

$= (4x^2+7)^8 + C$

**Example 1:** Evaluate the following integral:

$\int e^{x^3-1} \cdot 3x^2 \, dx = \int f(g(x)) \cdot g'(x) dx$ $f(x) = e^x$, $g(x) = x^3 - 1$ $= f(g(x)) + C = e^{x^3-1} + C$

$3x^2 = \frac{d}{dx}(x^3 - 1) = \frac{d}{dx}x^3 - \frac{d}{dx}1$

$= 3x^2 + 0 = 3x^2$

### General Indefinite Integral Formulas

1. $\int (f(x))^n \cdot f'(x) \, dx = \frac{(f(x))^{n+1}}{n+1} + C, n \neq -1$
2. $\int e^{f(x)} \cdot f'(x) \, dx = e^{f(x)} + C$ Ex 1
3. $\int \frac{1}{f(x)} \cdot f'(x) \, dx = \ln|f(x)| + C$

When it is not easy to recognize that the integrand is the result (or nearly the result) of the Chain Rule, the process of **substitution**, sometimes called ***u*-substitution**, can be used.

### Substitution Method:

1. Select $u$ (look for a more involved function where in the previous section we had just an $x$).

① $\int (\underbrace{\text{function}}_{u})^n \cdot \underbrace{\sim dx}_{du}$ ② $\int e^{\overset{\text{function}}{u}} \cdot \underbrace{\sim dx}_{du}$ Ex 1 ③ $\int \frac{1}{\underbrace{\text{function}}_{u}} \cdot \underbrace{\boxed{\sim dx}}_{du}$

2. Take the derivative of $u$ using the $\frac{du}{dx}$ notation.
3. Bring $dx$ to the right hand side of the equation.
4. Bring any constant multiples to the left-hand side of the equation.
5. Substitute to replace everything involving $x$ in the integral. You should now have a basic integral involving $u$'s.
6. Integrate (i.e., find the anti-derivative) your basic integral involving $u$'s.
7. In your result, replace all $u$'s with $x$'s. This is your answer!

$u = \text{function}$ $\left(\frac{du}{dx} = \text{function}'\right)$

$\Rightarrow du = \text{function}' \cdot dx$

**Example 2:** Let's use $u$-substitution for the integral from Example 1.

$\int e^{\overset{u}{x^3-1}} \cdot 3x^2 \, dx =$ Case 2

$u = x^3 - 1$

$\frac{du}{dx} = 3x^2$

$= \int e^u \cdot du$

$\Rightarrow du = 3x^2 dx$

$= e^u + C = \boxed{e^{x^3-1} + C}$

(3) Last 6.1

**Example 3:** Evaluate the following integrals:

a) $\int 7x(8x^2+3)^9\,dx$

$= \int 7\,u^9 \cdot \frac{1}{16}\,du = \frac{7}{16}\int u^9\,du$ (5) 6.1

diff $\boxed{u = 8x^2+3}$

multiply dx both side $\frac{du}{dx} = 2\cdot 8\cdot x + 0 = 16x$

$du = 16x\,dx$ divides by 16 for both side

$\boxed{\frac{1}{16}\,du = x\,dx}$

$= \frac{7}{16}\cdot\frac{1}{9+1}\cdot u^{10} + C$ (1) 6.1

$= \frac{7}{160}\,u^{10} + C$

$= \frac{7}{160}(8x^2+3)^{10} + C$

b) $\int \frac{12x}{3x^2+5}\,dx$

$= \int \frac{2}{u}\,du \overset{(5)\,6.1}{=} 2\int \frac{1}{u}\,du \overset{(4)\,6.1}{=} 2(\ln|u| + c)$

diff $\boxed{u = \text{denominator} = 3x^2+5}$ ②

$\frac{du}{dx} = 6x$

dx $du = 6x\,dx$

$\cdot 2$ $\boxed{2\,du = 2\cdot 6x\,dx = 12x\,dx}$ ①

$= 2\ln|u| + c$

② $\boxed{= 2\ln|3x^2+5| + c}$

$\int f u^n \sim dx$ or $\boxed{\int e^f \sim dx}$ or $\int \frac{n}{f}\cdot dx$

c) $\int \frac{2e^{5/x^4}}{3x^5}\,dx$

$= \int e^{\frac{5}{x^4}}\cdot\frac{2}{3x^5}\,dx \overset{(5)\,6.1}{=} \frac{2}{3}\int e^{\frac{5}{x^4}}\cdot x^{-5}\,dx$

diff $\boxed{u = \frac{5}{x^4} = 5\cdot x^{-4}}$

$\frac{du}{dx} = (-4)\cdot 5\cdot x^{-4-1} = -20\cdot x^{-5}$

dx $du = -20x^{-5}\cdot dx$

$\frac{1}{-20}$ $\boxed{-\frac{1}{20}\,du = x^{-5}\,dx}$

$= \frac{2}{3}\int e^u\left(-\frac{1}{20}\right)du$

$\overset{(5)\,6.1}{=} \frac{2}{3}\cdot\left(-\frac{1}{20}\right)\cdot\int e^u\,du = -\frac{1}{30}\cdot\int e^u\,du$

$\overset{(3)\,6.1}{=} -\frac{1}{30}(e^u + c)$

$= -\frac{1}{30}e^u + C$

$\boxed{= -\frac{1}{30}\cdot e^{\frac{5}{x^4}} + C}$

$-\frac{1}{7} + 1 = -\frac{1}{7} + \frac{7}{7} = \frac{6}{7}$ (6.1 / ①)

$\int u^n\,du \quad n \neq -1$

$= \frac{1}{n+1}\cdot u^{n+1} + C$

$n = -\frac{1}{7}$

$= \frac{1}{-\frac{1}{7}+1}\cdot u^{-\frac{1}{7}+1} + C$

$= \frac{1}{\frac{6}{7}}\cdot u^{\frac{6}{7}} + C$ $= \frac{7}{6}$

d) $\int \frac{8t^3}{\sqrt[7]{2-5t^4}}\,dt$

$t^3$: numerator

diff $\boxed{u = 2 - 5t^4}$

$\frac{du}{dx} = -5\cdot 4\,t^3 = -20\,t^3$

dx $du = -20\cdot t^3\cdot dt$ $\div -20$

$\boxed{-\frac{1}{20}\,du = t^3\,dt}$

$\int \frac{8t^3}{\sqrt[7]{2-5t^4}}\,dt \overset{(5)\,6.1}{=} 8\int \frac{t^3\,dt}{\sqrt[7]{2-5t^4}} = 8\int \frac{-\frac{1}{20}}{u^{\frac{1}{7}}}\,du$

$\sqrt[7]{u}$

$\overset{(5)\,6.1}{=} \frac{-8}{20}\cdot\int \frac{1}{u^{\frac{1}{7}}}\,du = -\frac{2}{5}\cdot\int u^{-\frac{1}{7}}\,du$

$\overset{(1)\,6.1}{=} -\frac{2}{5}\left(\frac{1}{-\frac{1}{7}+1}\cdot u^{-\frac{1}{7}+1} + c\right)$

$= -\frac{2}{5}\left(\frac{1}{\frac{6}{7}}\cdot u^{\frac{6}{7}} + c\right) = -\frac{2}{5}\cdot\frac{7}{6}\,u^{\frac{6}{7}} + c$

$= -\frac{7}{15}\cdot u^{\frac{6}{7}} + c = -\frac{7}{15}\cdot(2-5t^4)^{\frac{6}{7}} + C$

e) $\int \frac{1}{x\ln x}\,dx$

~~$u = x\ln x$~~ or $u = \ln x$

$= \int \frac{1}{\ln x}\cdot\left(\frac{1}{x}\cdot dx\right) = \int \frac{1}{u}\,du = \ln|u| + c$ (4) 6.1

diff $\boxed{u = \ln x}$

$\frac{du}{dx} = \frac{1}{x}$

dx $\boxed{du = \frac{1}{x}\cdot dx}$

$\boxed{= \ln|\ln x| + c}$

① $\int f^n \sim dx$ ② $\int e^f \cdot \sim dx$ ③ $\int \frac{\sim}{f} dx$

f) $\int \frac{2e^{8x}}{e^{8x}-7}dx = \int \frac{\left(\frac{1}{4}\right)}{u} du \overset{(5)}{=} \frac{1}{4} \int \frac{1}{u} du \overset{(4)}{=} \frac{1}{4} \ln|u| + c$ 6.1 6.1

$\cancel{u=8x}$ $\boxed{u = e^{8x}-7}$

$= \frac{1}{4} \ln|e^{8x}-7| + c$

$\frac{d(e^{8x})}{dx} = e^{8x} \cdot (8x)' = 8e^{8x}$

$\frac{du}{dx} = 8e^{8x}$

$du = 8e^{8x}dx$ ) divides both side by 4

$\boxed{\frac{1}{4}du = 2e^{8x}dx}$

g) $\int (3x^4+6)(10x+x^5+7)^6 dx = \int 3(x^4+2)(10x+x^5+7)^6 dx = \int 3 \cdot \frac{u^6}{5} du$

$\boxed{u = 10x + x^5 + 7}$ $\boxed{\int f^n \sim dx \text{ form}}$

$\overset{(5)}{=} \frac{3}{5} \int u^6 du$ 6.1

diff( $\frac{du}{dx} = 10 + 5x^4 = 5(x^4+2)$ $\sim$ $(3x^4+6) = 3(x^4+2)$

$\overset{(1)}{=} \frac{3}{5}\left(\frac{1}{6+1}u^{6+1} + c\right)$ 6.1

dx( $du = 5(x^4+2)dx$

$\frac{1}{5}$( $\boxed{\frac{1}{5}du = (x^4+2)dx}$

$= \frac{3}{5} \cdot \frac{1}{7} \cdot u^7 + C$

$= \frac{3}{35}u^7 + C$

$= \frac{3}{35}(10x+x^5+7)^7 + C$

h) $\int (25+4e^{5x-7})\ dx$

$\overset{(6)}{=} \int 25dx + \int 4e^{5x-7}dx \overset{(5)}{=} 25x + c + 4\int e^{5x-7}dx \boxed{= 25x + \frac{4}{5}e^u + C}$ 6.1 6.1

$\int e^{5x-7}dx = \int e^u \cdot \frac{1}{5}du \overset{(5)}{=} \frac{1}{5} \cdot \int e^u du \overset{(3)}{=} \boxed{\frac{1}{5} \cdot e^u + c}$ 6.1 6.1

$\boxed{u = 5x-7}$

$\frac{du}{dx} = 5$

$du = 5dx \Rightarrow \boxed{\frac{1}{5}du = dx}$

**Example 4:** Find $f(x)$ if $\boxed{f(2)=0}$ and $f'(x) = \frac{x^4}{x^5+1}$.

$\int f'(x)\,dx = \int \frac{x^4}{x^5+1}dx = \int \frac{1}{u} \cdot \frac{1}{5}du \overset{(5)}{=} \frac{1}{5}\int \frac{1}{u} du$ 6.1

$\boxed{f(x) = \frac{1}{5}\ln(x^5+1) - \frac{1}{5}\ln 33}$

$\boxed{u = x^5+1}$

$\overset{(4)}{=} \frac{1}{5}\ln|u| + C$ 6.1

$\frac{du}{dx} = 5x^4$

$du = 5x^4 dx$

$\frac{1}{5}$( $\boxed{\frac{1}{5}du = x^4 dx}$

$Y(x) = \frac{1}{5}\ln(x^5+1) + C$

If $Y(x) = f(x)$
$\Rightarrow Y(2) = f(2) = 0$

$\frac{1}{5} \cdot \ln(2^5+1) + C = 0$

$C = -\frac{1}{5}\ln 33$