Section 6.1: Antiderivatives

Antidifferentiation - Reconstructing a function from its derivative.

*A function F is an **antiderivative** of a function f if $F'(x) = f(x)$.

Example 1: Find three antiderivatives of $f(x) = x$.

$\int f(x)\,dx = \frac{1}{2}x^2 + C$

① $\frac{1}{2}x^2 \longrightarrow \frac{d}{dx}\left(\frac{1}{2}\cdot x^2\right) = \frac{1}{2}\cdot(2x) = x$

② $\frac{1}{2}x^2 + 4 \to \frac{d}{dx}\left(\frac{1}{2}x^2+4\right) = \frac{d}{dx}\left(\frac{1}{2}x^2\right) + \frac{d}{dx}(4)$

$= x + 0$

$= x$

$\Rightarrow \boxed{\frac{1}{2}x^2 + C}$ ← constant works as an antiderivative.

*Thus, antidifferentiation leads not to a unique function, but to an entire **family** of functions (antiderivatives).

Indefinite Integrals - We use the symbol

integral sign, integrand, differential

$$\boxed{\int f(x)\,dx}$$

$\int f(x)\,dx$: antiderivative of $f(x)$ w.r.t variable x.

called the **indefinite integral**, to represent the family of antiderivatives of $f(x)$, and we write

$$\int f(x)\,dx = F(x) + C$$

constant (real #)

if $F'(x) = f(x)$.

*The symbol $\int$ is called an **integral sign**, and the function $f(x)$ is called the **integrand**. The symbol dx indicates that antidifferentiation is performed with respect to the variable x. $F(x)$ is the antiderivative of $f(x)$, and the arbitrary constant C is called the **constant of integration**.

Don't forget to constants!

*Thus, for the above example, we could write $\int x\,dx = \frac{1}{2}x^2 + C$.

Formulas and Properties for Integration - For constants C and k,

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, where $n \neq -1$ $\qquad \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = \frac{n+1}{n+1}\cdot x^n = x^n$
2. $\int k\,dx = kx + C$ $\qquad k \in \mathbb{R}$, $\qquad \frac{d}{dx}(kx+C) = k+0 = k$
3. $\int e^x\,dx = e^x + C$.
4. $\int \frac{1}{x}\,dx = \ln|x| + C$, where $\boxed{x \neq 0}$ $\qquad \frac{1}{x} = x^{-1}$ $\qquad \frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$
5. $\int kf(x)\,dx = k\int f(x)\,dx$ $\qquad k \in \mathbb{R}$ $\qquad \frac{d}{dx}(k\cdot F(x)) = k\cdot\frac{d}{dx}F(x)$
6. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example 2: Find the following indefinite integrals (i.e., the most general antiderivative)

a) $\int 8\,dx \overset{(2)}{=} 8x + C$

$\frac{d}{dx}(8x+C) = \frac{d}{dx}(8x) + \frac{d}{dx}C = 8 + 0 = 8$

b) $\int x^5\,dx \overset{(1)}{=} \frac{1}{5+1}\cdot x^{5+1} + C = \frac{1}{6}\cdot x^6 + C$

$\frac{d}{dx}\left(\frac{1}{6}\cdot x^6 + C\right) = \frac{6}{6}x^5 + 0 = x^5$

c) $\int \frac{1}{3}x^4\,dx \overset{(5)}{=} \frac{1}{3}\cdot\int x^4\,dx \overset{(1)}{=} \frac{1}{3}\cdot\left(\frac{1}{4+1}\cdot x^{4+1} + C\right)$

$= \frac{1}{3}\cdot\left(\frac{1}{5}\cdot x^5 + C\right) = \frac{1}{15}\cdot x^5 + \frac{1}{3}\cdot C$

$= \frac{1}{15}x^5 + C$

d) $\int \frac{3}{5x^2}\,dx = \int \frac{3}{5}x^{-2}\,dx$

$\overset{(5)}{=} \frac{3}{5}\cdot\int x^{-2}\,dx \overset{(1)}{=} \frac{3}{5}\cdot\left(\frac{1}{-2+1}\cdot x^{-2+1} + C\right)$

$= \frac{3}{5}\cdot\left(-x^{-1} + C\right) = -\frac{3}{5}x^{-1} + C$

e) $\int (5x^4 + x^3 - 2)\,dx$

$\overset{(6)}{=} \int 5x^4\,dx + \int x^3\,dx + \int (-2)\,dx$

$\downarrow(2) \quad \downarrow(1) \quad \downarrow(2)$

$= 5\cdot\int x^4\,dx + \frac{1}{3+1}\cdot x^{3+1} + C + (-2)x + C$

$= 5\int x^4\,dx + \frac{1}{4}\cdot x^4 - 2x + C$

$\overset{(1)}{=} 5\left(\frac{1}{4+1}\cdot x^{4+1}\right) + C + \frac{1}{4}x^4 - 2x + C$

$= \frac{5}{5}\cdot x^5 + \frac{1}{4}x^4 - 2x + C$

$= x^5 + \frac{1}{4}\cdot x^4 - 2x + C$

f) $\int \frac{3}{x}\,dx$ $\quad \boxed{x^{-1} = \frac{1}{x}}$

$\overset{(5)}{=} 3\cdot\int x^{-1}\,dx$

$\overset{(4)}{=} 3\cdot(\ln|x| + C)$

$= 3\ln|x| + C$

g) $\int \left(3\sqrt{x} - \frac{1}{x^2} - x^{3/2} + 4e^x\right)dx \overset{(6)}{=} \int 3\sqrt{x}\,dx + \int -\frac{1}{x^2}\,dx + \int -x^{\frac{3}{2}}\,dx + \int 4e^x\,dx$

$\overset{(5)}{=} 3\int x^{\frac{1}{2}}\,dx - \int x^{-2}\,dx - \int x^{\frac{3}{2}}\,dx + 4\int e^x\,dx$

$\downarrow(1) \quad \downarrow(1) \quad ③ \quad (3)$

$= 3\cdot\left(\frac{1}{\frac{1}{2}+1}\cdot x^{\frac{1}{2}+1} + C\right) - \left(\frac{1}{-2+1}x^{-2+1} + C\right)$

③: $-\left(\frac{1}{\frac{3}{2}+1}\cdot x^{\frac{3}{2}+1} + C\right) + 4\cdot(e^x + C)$

$= \cancel{3}\cdot\frac{2}{\cancel{3}}\cdot x^{\frac{3}{2}} + x^{-1} - \frac{2}{5}\cdot x^{\frac{5}{2}} + 4e^x + C$

h) $\int \left(\frac{6}{x}+\frac{5}{2x^4}-\frac{x^6}{3}\right)\,dx \overset{(6)}{=} \int \frac{6}{x}dx + \int \frac{5}{2x^4}dx + \int \frac{x^6}{3}dx$

$\overset{(5)}{=} 6\int x^{-1}dx + \frac{5}{2}\int x^{-4}dx - \frac{1}{3}\int x^6 dx$

$\downarrow(4) \qquad \downarrow(1) \qquad (1)$

$= 6(\ln|x|+c) + \frac{5}{2}\cdot\left(\frac{1}{-4+1}\cdot x^{-4+1}+c\right) + \frac{1}{3}\cdot\left(\frac{1}{6+1}\cdot x^{6+1}+C\right)$

$= 6\ln|x| + \frac{5}{2}\cdot\frac{1}{-3}\cdot x^{-3} + \frac{1}{3}\cdot\frac{1}{7}\cdot x^7 + C$

$= 6\ln|x| - \frac{5}{6}x^{-3} + \frac{1}{21}x^7 + C$

i) $\int \frac{4u+u^3-3u^{-7}}{5u^2}\,du$

$= \int\left(\frac{4u}{5u^2}+\frac{u^3}{5u^2}-\frac{3u^{-7}}{5u^2}\right)du = \int\left(\frac{4}{5}u^{-1}+\frac{u}{5}-\frac{3}{5}u^{-9}\right)du$

$\overset{(6)}{=} \int \frac{4}{5}u^{-1}du + \int \frac{u}{5}du + \int\left(-\frac{3}{5}\right)u^{-9}du \overset{(5)}{=} \frac{4}{5}\int u^{-1}du + \frac{1}{5}\int u\,du - \frac{3}{5}\int u^{-9}du$

$\swarrow(4) \qquad \downarrow(1) \qquad \downarrow(1)$

$= \frac{4}{5}(\ln|u|+c) + \frac{1}{5}(u^2+c) - \frac{3}{5}\cdot\frac{1}{-9+1}\cdot u^{-9+1}$

$= \frac{4}{5}\ln|u| + \frac{1}{5}u^2 + \frac{3}{40}u^{-8} + C$

j) $\int x(x^2+2)\,dx = \int(x^3+2x)dx$

$\overset{(6)}{=} \int x^3dx + \int 2x\,dx$

$\downarrow(5)$

$=\quad 〃 \quad + 2\int x\,dx$

$\overset{(1)}{=} \frac{1}{3+1}\cdot x^{3+1}+c + 2\left(\frac{1}{1+1}\cdot x^{1+1}+c\right)$

$= \frac{1}{4}\cdot x^4 + x^2 + C$

k) $\int (x-2)(x+3)\,dx = \int(x^2-2x+3x-6)dx$

$= \int(x^2+x-6)dx$

$\overset{(6)}{=} \int x^2dx + \int x\,dx + \int(-6)dx$

$\downarrow(1) \qquad \downarrow(1) \qquad \searrow(2)$

$= \frac{1}{2+1}x^{2+1}+c + \frac{1}{1+1}\cdot x^{1+1}+c - 6x+c$

$= \frac{1}{3}\cdot x^3 + \frac{1}{2}\cdot x^2 - 6x + C$

l) $\int \frac{9-e^{-x}}{2e^{-x}}\,dx = \int\left(\frac{9}{2e^{-x}} - \frac{\cancel{e^{-x}}}{2\cancel{e^{-x}}}\right)dx$

$\left(\frac{1}{e^{-x}} = e^{-(-x)} = e^x\right) = \int\left(\frac{9}{2}\cdot e^x - \frac{1}{2}\right)dx$

$\overset{(6)}{=} \frac{9}{2}\int e^x dx + \int\left(-\frac{1}{2}\right)dx$

$\downarrow(3) \qquad \downarrow(2)$

$= \frac{9}{2}(e^x+c) + \left(-\frac{1}{2}x+c\right)$

$= \frac{9}{2}e^x - \frac{1}{2}x + C$

boundary condition $y'(t)$

Example 3: Find $y(t)$ if $\boxed{y(1) = 1}$ and $\frac{dy}{dt} = \frac{3}{t} + \frac{1}{t^2}$.

$Y(t) = \int y'(t)\,dt = \int \left(\frac{3}{t} + \frac{1}{t^2}\right) dt$

(6) $= \int \frac{3}{t}\,dt + \int \frac{1}{t^2}\,dt$

(5) $= 3 \cdot \int t^{-1}\,dt + \int t^{-2}\,dt$

(4) (1)

$= 3(\ln|t| + C) + \frac{1}{-2+1} \cdot t^{-2+1} + C$

$= 3\ln|t| - t^{-1} + C$

If $Y(t) = y(t)$ $\Rightarrow Y(1) = y(1) = 1$

$y(1) = 3 \cdot \ln 1 - \frac{1}{1} + C = -1 + C \Rightarrow C = 2$

$y(t) = 3\ln|t| - t^{-1} + 2$

Procedure

(1) Find out Antiderivative

Call this as $Y(t)$

(2) Plug in the condition to get $y(t)$

Example 4: The $\boxed{\text{marginal revenue}}$ of selling x watches each day is given by $\boxed{R'(x)} = 30 - 0.0003x^2$ dollars per watch for $0 \le x \le 540$. If the revenue is \$1487.50 when 50 watches are sold, find the revenue function.

① $R(50) = 1487.50$

(1) $\int R'(x)\,dx = \int (30 - 0.0003x^2)\,dx \overset{(6)}{=} \int 30\,dx + \int -0.0003x^2\,dx$

(2) (5)

$= 30x + C - 0.0003 \int x^2\,dx$

(1)

$= 30x + C - 0.0003\left(\frac{1}{2+1}x^{2+1} + C\right)$

$Y(t) = 30x - 0.0001x^3 + C$

(2) plug in 50 into Y

$1487.5 = Y(50) = 30 \cdot 50 - 0.0001 \cdot 50^3 + C = 1487.50 + C$

$\Rightarrow C = 0$

$R(x) = 30x - 0.0001x^3 + 0$

Example 5: A sculpture purchased by a museum for \$50,000 increases in value at a rate of $V'(t) = 100e^t$ dollars per year, where t is the time in years since the purchase. What will the sculpture be worth in 12 years?

(1) $Y(t) = \int 100e^t \overset{(5)}{=} 100 \int e^t = 100e^t + C$

(2) Plug in $t = 0$ $\quad 50{,}000 = Y(0) = 100 \cdot e^0 + C = 100 + C$

$\Rightarrow C = 49{,}900$

because when $t=0$ (present) sculpture has value \$50000.

$\Rightarrow V(t) = 100e^t + 49{,}900$

$V(12) = 100e^{12} + 49{,}900 = \$16{,}325{,}379.14$