Math 142 Fall 2019 - ~~Week 3~~ - Review-Midterm 2

~~2/5/2020~~ -
3/24/2020

Here's what you need to know to get the perfect grade.

$\frac{df}{dx} = f'(x)$

ex) $f(x) = \ln(x^2+1)$, $f(x) = g(h(x))$, $g(x) = \ln x$, $g'(x) = \frac{1}{x}$, $h(x) = x^2+1$, $h'(x) = 2x$

(1) 4.3-4.4 Chain Rule.

(a) Think a complicated function as a composition of simple functions.

(b) For given $f(x) = g(h(x))$,

$$① \ \boxed{g'(h(x))h'(x) = f'(x)} = ② \ \boxed{\frac{df}{dx} = \frac{dg}{du}\cdot\frac{dh}{dx}}.$$

$\frac{d(x^2+1)}{dx} = 2x$

In other words, Leibniz Notation and Newton notation are just the same thing.

$f'(x) = g'(h(x))\cdot h'(x) = \frac{1}{\ln 10}\cdot\frac{1}{5^{x^2-1}}\cdot \ln 5\cdot 5^{x^2-1}\cdot(2x)$

$h(x) = 5^{x^2-1}$

(c) Ex) $f(x) = \log(5^{x^2-1})$. Find $f'(x)$.

$f'(x) = g'(h(x))\cdot h'(x)$ — chain rule

$g(x) = \log x = \log_{10} x$

$h(x) = 5^{x^2-1}$, $h'(x) = \ln 5\cdot 5^{x^2-1}\cdot 2x$, $g'(x) = \frac{1}{\ln 10}\cdot\frac{1}{x}$

$q(x) = 5^x$, $j(x) = x^2-1$, $h(x) = q(j(x))$, $h'(x) = q'(j(x))\cdot j'(x) = \ln 5\cdot 5^{x^2-1}\cdot 2x$

$\frac{dg}{du} = \frac{1}{u} = \frac{1}{x^2+1}$, $g(u)$, ex) $g(u) = \ln u$, $u = h(x)$

$\frac{dg}{du} = g' = \frac{1}{u} = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}$

(2) 5.1 The First Derivative

(a) $f'(x) > 0$ (positive) on an interval $\iff$ f is increasing on interval.

(b) $f'(x) < 0$ (negative) on an interval $\iff$ f is decreasing on interval.

(c) $f(x)$ has a *local maxima* at $x = c$ if $f(c) \geq f(x)$ when x is near c.
some point in domain of f.

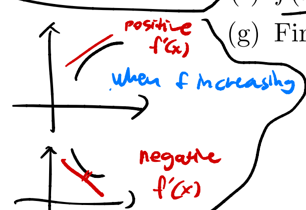
(d) $f(x)$ has a *local minima* at $x = c$ if $f(c) \leq f(x)$ when x is near c.

a: local max, b: local min.

(e) $f(x)$ has a *local extrema* at $x = c$ if f has local maxima or minima at $x = c$.

(f) $f(x)$ has a *critical value* at $x = c$ if 1) $f(c)$ exists 2) $f'(c) = 0$ or does not exist.
$c \in$ domain of f

(g) First derivative test



- Assumption: $f(x)$ is a continuous function on an interval (a, b), and $c \in (a, b)$ is a critical value of f. $c \in$ dom f
- Conclusion 1: If sign of $f'(x)$ changes from + (positive) to - (negative) at $x = c$, then f has a local maxima at c.
- Conclusion 2: If sign of $f'(x)$ changes from - (negative) to + (positive) at $x = c$, then f has a local minima at c.
- Conclusion 3: If sign of $f'(x)$ does not change at $x = c$, then f has no local minima or local maxima at c.

(h) Ex) Find local extremas of $f(x) = 8\ln x - x^2$ using the first derivative test.
domain $(0, \infty)$

① Find $f'(x)$: $f'(x) = \frac{8}{x} - 2x$

② Find critical values: $x = 0 \Rightarrow f'(x)$ DNE; $f'(x) = 0 \Rightarrow \frac{8}{x} = 2x \Rightarrow 8 = 2x^2 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2$

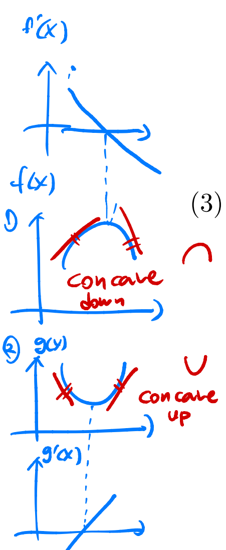
critical value $\boxed{x = 2}$, -2, 0 not critical value. $\ln 0$, $\ln -2$ not defined.

③ First derivative test

Sign $f'(x)$: 0 ⊕ 2 ⊖; if $x \in (0,2)$, $x \in (2,\infty)$; ex) $f'(1) = \frac{8}{1} - 2\cdot 1 > +$, $f'(3) = \frac{8}{3} - 2\cdot 3 < -$

④ Conclusion: since sign changes from + to − at $x=2$ ⇒ f has local maxima at $x=2$

(3) 5.2 The Second Derivative



(a) $f''(x) = \frac{d^2 f}{dx^2}$; this is just notation.

(b) f is *concave upward* on an interval (a, b) $\iff$ $f'(x)$ is increasing on (a, b) $\iff$ $f''(x) > 0$ (positive) on an interval (a, b).

(c) f is *concave downward* on an interval (a, b) $\iff$ $f'(x)$ is decreasing on (a, b) $\iff$ $f''(x) < 0$ (negative) on an interval (a, b).

c is partition number

(d) $f(x)$ has an $\boxed{\textit{inflection point}}$ at $x = c$ if $\boxed{\text{1) } f(c) \text{ exists 2) } f''(c) = 0 \text{ or does not exist}}$ 3) sign of $f''(x)$ changes.

ex) Find interval where $f(x) = 8\ln x - x^2$ is concave up: $f''(x) = -\frac{8}{x^2} - 2 = 0 \Rightarrow -8 = 2x^2$ since $x^2 > 0$ there is no solution ⇒ $x = 0$ is possible partition number ⇒ $0 \notin$ dom f ⇒ No partition #.

(e) Second derivative test

partition number: some number in domain of f s.t. $f''(x) = 0$ or DNE

- Assumption: $f(x)$ is a continuous function on an interval (a, b), and $c \in (a, b)$ is a critical value of f such that $\boxed{f'(c) = 0}$.

Draw sign chart of $f''(x)$

f is concave down for $(0, \infty)$ ⇐ f is concave down; 0 ⊖ ∞; $x=2$: $f''(2) = -\frac{8}{4} - 2 < 0$

at $x=c$

- Conclusion 1: If $f''(c) > 0$ (positive), then f has a local minima at c.
- Conclusion 1: If $f''(c) < 0$ (negative), then f has a local maxima at c.
  c at $x=c$
- Note: If c is a critical value but $f'(c)$ does not exist, then we cannot use the second derivative test.

at $x=3$ f has local maxi; at $x=-1$: f has local min.

(f) Ex) Find local extremas of $f(x) = 1 + 9x + 3x^2 - x^3$ using Second Derivative Test.

ex)

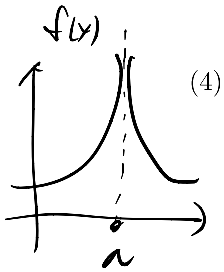
① $f'(x) = 9 + 6x - 3x^2 \Rightarrow$ critical value $f(x) = 3(-x^2+2x+3) = 3(x+1)(-x+3)$

$x = -1$ or 3

② $f''(x) = 6 - 6x$

plug in at $f''(x)$

$f''(-1) = 12 > 0$, $f''(3) = -12 < 0$



vertical asymptote at $x=a$.

(4) 5.3 Limits at infinity.

(a) Given a curve $y = f(x)$, *vertical asymptote* is a line $x = a$ if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

(b) Difference between hole and vertical asymptote. ex) $f(x) = \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x-2)} \Longrightarrow$ hole at $x = -1$, $VA = 2$.

(c) Given a curve $y = f(x)$, *horizontal asymptote* is a line $y = L$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$. (So there are at most two horizontal asymptotes.)

(d) Find horizontal asymptote for rational function: Divide every term by the highest degree part of x in *denominator*. For example, $f(x) = \frac{2x^2+1}{x^4+x-2}$

$a\ x^{-2} = \frac{1}{x^2}$

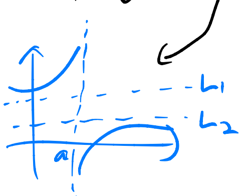
$$\lim_{x\to\infty} \frac{2x^2+1}{x^4+x-2} = \lim_{x\to\infty} \frac{2x^{-2}+x^{-4}}{1+x^{-3}-2x^{-4}} = \lim_{x\to\infty} \frac{0+0}{1+0-0} = 0.$$

Horizontal asymptote at $y=0$

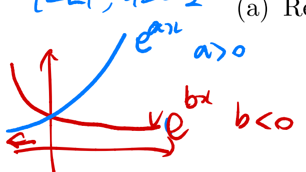
(e) Find horizontal asymptote for rational function containing exponential term: Divide every term by the term of e^x which is *furthest* from zero. For example,

$$\lim_{x\to\infty} \frac{4e^{2x}-6e^{-3x}}{7+e^{2x}+e^{-6x}+e^{-8x}} = \lim_{x\to\infty} \frac{4-6e^{-5x}}{7e^{-2x}+1+e^{-8x}+e^{-10x}} = \frac{4-0}{0+1+0+0} = 4.$$

$$\lim_{x\to-\infty} \frac{4e^{2x}-6e^{-3x}}{7+e^{2x}+e^{-6x}+e^{-8x}} = \lim_{x\to-\infty} \frac{4e^{10x}-6e^{5x}}{7e^{8x}+e^{10x}+e^{2x}+1} = \frac{0-0}{0+0+0+1} = 0.$$



Horizontal asymptote $y=L_1$, $y=L_2$



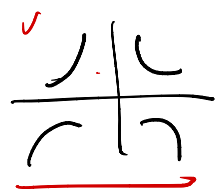
e^{ax} $a>0$, e^{bx} $b<0$

If $x\to\infty$ e^{ax} is the furthest term from 0 if $a>0$ a is greater than any of other coefficients b of e^{bx}.

(5) 5.4 Graph Sketching

(a) Remember the table.

	$f'(x) > 0$	$f'(x) < 0$
$f''(x) > 0$	Increasing Concave Up	Decreasing Concave Up
$f''(x) < 0$	Increasing Concave Down	Decreasing Concave Down



(b) Critical points give local maxima or local minima generally.

(c) Use information of vertical and horizontal asymptote.

(d) Ex) Draw a graph from given information.

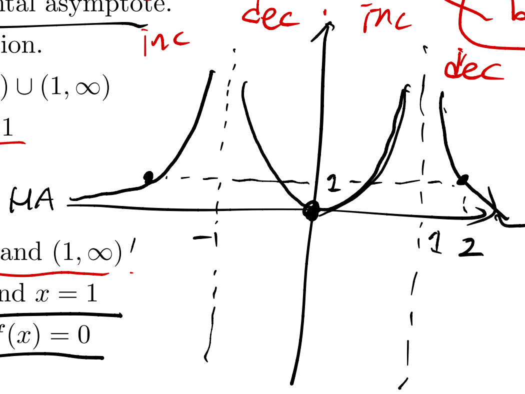
- Domain of $f : (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
- $f(-2) = 1, f(0) = 0$, and $f(2) = 1$
- $f'(x) > 0$ on $(-\infty, -1)$ and $(0,1)$
- $f'(x) < 0$ on (-1,0) and $(1, \infty)$
- $f''(x) > 0$ on $(-\infty, -1), (-1, 1)$, and $(1, \infty)$
- Vertical asymptotes at $x = -1$ and $x = 1$
- $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$

Concave up

dec | inc

inc

dec



If $x\to-\infty$ e^{ax} is the furthest term from 0 if $a<0$ and a is lower than any of other coefficient b of e^{bx}

(6) 5.5 Absolute Maxima and minima

(a) Assumption: $f(x)$ is a continuous function on a *closed* interval $[a, b]$. Let $c \in [a, b]$

(b) Conclusion 1: $f(x)$ has the *absolute maxima* at $x = c$ if $f(c) \geq f(x)$ for all $x \in [a, b]$

(c) Conclusion 1: $f(x)$ has the *absolute minima* at $x = c$ if $f(c) \leq f(x)$ for all $x \in [a, b]$

(d) How to find:

(i) Find all critical values of $f(x)$ inside of $[a, b]$.

(ii) Compare $f(x)$ at $x = a$, $x = b$, and $x =$ critical values. Find x giving the greatest (resp. the lowest) $f(x)$ among $x = a$, $x = b$, and $x =$ critical values. That x is the absolute maxima (resp. minima).

(iii) Ex) $f(x) = x^3 - 3x + 5$. Find the absolute maxima and absolute minima on $[0, 3]$.

① $f'(x) = 3x^2 - 3$ poly $[0, 3]$

$= 3(x^2 - 1)$

$f'(x) = 0$ when $x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

$\Rightarrow$ f has critical value at $x = 1$ on $[0, 3]$

② $f(0) = 0 - 0 + 5 = 5$

$f(1) = 1 - 3 + 5 = 3$ lowest.

$f(3) = 27 - 9 + 5 = 23$ greatest

$\Rightarrow$ f has the abs. max at $x = 3$

f has " min at $x = 1$.