# 142_12c_5.6

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## 5.6 Supplement: Optimization

### Strategy for Solving Optimization Problems

1. Introduce variables, look for relationships among the variables, and construct a mathematical model of the form

"Maximize (or minimize) $f(x)$"

2. Determine the interval (domain) of the model (function) you created.
3. Use optimization techniques from calculus to find a solution:
   - Find the critical values of $f(x)$.
   - Use the procedures developed in Section 5.5 to find the absolute maximum (or minimum) value of $f(x)$ on the interval $I$ and the value(s) of $x$ where this occurs.

4. Answer all the questions asked in the problem.

**Example:** Find two (non-negative) numbers $x$ and $y$ such that $2x + y = 34$, and the product of the numbers is a maximum.

① Find $x, y$ s.t.
1) $x \ge 0,\ y \ge 0$
2) $2x + y = 34$
3) $x \cdot y$ is the maximum among all $x, y$ satisfying 1) and 2).

Maximize $x \cdot y$
subject to $2x + y = 34 \Rightarrow y = 34 - 2x$
$x \ge 0,\ y \ge 0$

Maximize $x \cdot (34 - 2x) = 34x - 2x^2$
s.t. $x \ge 0$ $\quad 34 - 2x \ge 0$
$34 \ge 2x$
$17 \ge x$

Maximize $x(34 - 2x)$
s.t. $17 \ge x \ge 0$

$\Rightarrow$ ( $f(x) = x(34 - 2x)$. Find absolute Max on $[0, 17]$ )

$\Rightarrow f'(x) = (34 - 2x) \cdot 1 + x(-2)$
$= 34 - 2x - 2x = 34 - 4x$
$f'(x) = 0$ when $34 - 4x = 0 \Rightarrow \boxed{x = \frac{34}{4}}$ critical value since $\frac{34}{4} \in$ dom $f$.

$f(0) = 0 \cdot (34 - 2 \cdot 0) = 0$
$f(17) = 17(34 - 2 \cdot 17) = 0$
$f(\frac{34}{4}) = \frac{34}{4}(34 - 2 \cdot \frac{34}{4})$
$= \frac{17}{2} \cdot (17) = \frac{(17)^2}{2} > 0$

So $f$ has abs. max at $x = \frac{34}{4}$.

$x = \frac{34}{4} = \frac{17}{2}$
$y = 34 - 2 \cdot \frac{34}{4}$
$= 34 - \frac{34}{2}$
$= 34 - 17$
$= 17.$



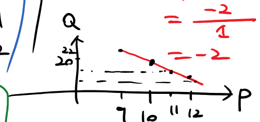
**Example:** Suzie can sell 20 bracelets each day when the price is \$10 for a bracelet. If she raises the price by \$1, then she sells 2 fewer bracelets each day. If it costs \$8 to make each bracelet, find the selling price that will maximize Suzie's profit.

$R(x) = (\text{price} - \text{Cost}) \cdot \text{quantity} = \text{Price} \cdot \text{Quantity} - \text{total cost}.$

| Q | P |
|---|---|
| 20 | \$10 |
| 18 | \$11 |
| 16 | \$12 |

$Q = Q(P)$

* Slope of $Q(P)$
$= \frac{-2}{1}$
$= -2$



$Q = -2(P - 10) + 20$
$\boxed{Q = -2P + 40}$
$= -2P + 40$

$R(P) = (P - 8)\,Q(P)$
$= (P - 8)(-2P + 40)$
$= -2P^2 + 16P + 40P - 320$
dom $R(P) = \boxed{[0, \infty)}$
$R'(x) = -4P + 56$
Critical values.
$R'(x) = 0$ or DNE.
$-4P + 56 = 0 \Rightarrow 4P = 56 \Rightarrow \boxed{P = 14}$ ← Critical Value

$R(14) = (14 - 8)(-28 + 40)$
$= 6 \cdot 12 = 72\$$
$R(0) = 0$
Selling price = 14\$

$xy = 1400$

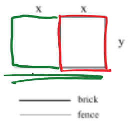
**Example:** Ben needs to enclose two rectangular regions that share one side, and each has an area of 1400 ft². One of the regions needs to be enclosed on all four sides by a brick wall, and the other region just needs fencing on its remaining three sides (see diagram below). The brick wall costs \$18 per foot, and the fencing costs \$6 per foot. Find the dimensions of each region that would be the most economical for Ben.

x x

$C(x) = (2x + 2y) \cdot 18 + (2x + y) \cdot 6$

$x+y = 1400$

**Example:** Ben needs to enclose two rectangular regions that share one side, and each has an area of 1400 ft². One of the regions needs to be enclosed on all four sides by a brick wall, and the other region just needs fencing on its remaining three sides (see diagram below). The brick wall costs $18 per foot, and the fencing costs $6 per foot. Find the dimensions of each region that would be the most economical for Ben.




$$\boxed{\begin{aligned} &C(x) = (2x+2y)\cdot 18 + (2x+y)\cdot 6 \\ &\text{Minimize } C(x). \\ &\text{s.t. } (\cancel{2}x)\cdot y = 1400 \Rightarrow y = \frac{1400}{\cancel{2}x} = \frac{1400}{x} \end{aligned}}$$

$$C(x) = \left(2x + 2\cdot\frac{1400}{x}\right)\cdot 18 + \left(2x + \frac{1400}{x}\right)\cdot 6$$

$$= 36x + \frac{1400\cdot 36}{x} + 12x + \frac{6\cdot 1400}{x}$$

$$= 48x + \frac{1400}{x}(36+6)$$

$$= 48x + \frac{42\cdot 1400}{x}$$

Minimize $C(x) = 48x + \frac{42\cdot 1400}{x}$

s.t. $x \geq 0$. $(0,\infty)$

$\Rightarrow C'(x) = 48 - 42\cdot 1400\cdot x^{-2}$

Find $x$ where $C'(x)=0$ or DNE $\Rightarrow 48 - 42\cdot 1400\cdot x^{-2} = 0$ move

$48x^2 = 42\cdot 1400$ $\quad 48 = 21\cdot 1400 x^{-2}\cdot x^2$

$x^2 = \frac{42\cdot 1400}{48}$

$x = \pm\frac{\sqrt{42\cdot 1400}}{\sqrt{48}}$

$= \pm\frac{\sqrt{3}\cdot 140}{\sqrt{3}\cdot 4} = \pm\frac{140}{4} = \pm 35$

$\sqrt{48} = \sqrt{3\cdot 16} = \sqrt{3}\cdot\sqrt{16} = 4\sqrt{3}$

$42 = 3\cdot 7\cdot 2$ $\quad 48 = 2^4\times 3$

$1400 = 2\cdot 7\cdot 10\cdot 10$

$42\cdot 1400 = 3\cdot 2^2\cdot 7^2\cdot 10^2$

$\sqrt{\ \ }\ // \ = \sqrt{3}\cdot 2\cdot 7\cdot 10$

$x = \frac{35}{\sqrt{2}}$

$y = \frac{1400}{2\cdot\frac{35}{\sqrt{2}}} = \ldots$

Critical value $= +35$

(because $-35$ is not in the domain)

$C''(35) = 84\cdot 1400\cdot\frac{1}{\left(\frac{35}{\sqrt{2}}\right)^3} > 0$

$\Rightarrow$ By second derivative test, $C$ has local min on $x=35$

$C''(x) = -21\cdot 1400\cdot x^{-3}(-2)$

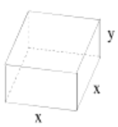
$= 42\cdot 1400\, x^{-3}$

$\Rightarrow$ Since $x=35$ is the only critical value and there is no boundary points, it is absolute minima.





**Example:** You need to create a box with a square base and an open top that has the largest volume possible (see diagram below). You have 1200 cm² of material to make the box. Find the dimensions and volume of your box.




$$\boxed{x^2 + (x\cdot y)\cdot 4 = 1200}$$

$x^2 + 4xy = 1200$

Volume $V = x\cdot x\cdot y$

Maximize $V(x) = x^2 y$

s.t. $\boxed{x^2 + 4xy = 1200}$

$\Rightarrow 4xy = 1200 - x^2$

$y = \frac{1200}{4x} - \frac{x^2}{4x}$

$= \frac{300}{x} - \frac{x}{4}$.

Maximize $V(x) = x^2\cdot\left(\frac{300}{x} - \frac{x}{4}\right)$

s.t. $x \geq 0$ domain of $V(x) = (0,\infty)$

$V(x) = 300x - \frac{x^3}{4}$

$V' = 300 - \frac{3x^2}{4} = 0$

when $300 = \frac{3x^2}{4} \Rightarrow 1200 = 3x^2$

$400 = x^2$

$x = \pm 20$

$x = 20$

$V'' = -\frac{6}{4}x$

$V''(20) = -\frac{6}{4}\cdot 20 < 0$

$\Rightarrow x=20$ gives local max.

$\Rightarrow$ Abs max

$x = 20$

$y = \frac{300}{20} - \frac{20}{4}$

$= 15 - 5$

$= 10$

**Example:** College Park Apartments can rent 250 apartments when the rent is $650 each month. For each $10 increase in the rent, 2 additional apartments are left unoccupied. What monthly rent should the apartment complex charge to maximize the total rent collected? What is the maximum total rent?

Sol) Maximize total rent.

| Q | P |
|---|---|
| 250 ← | 650$ |
| 248 ← | 660$ |
| 246 ← | 670$ |
| ⋮ | |

$\Rightarrow \frac{-2}{10}$ is the slope –

$Q(P) = -\frac{2}{10}(P - 650) + 250$

$= -\frac{2}{10}P + 130 + 250$

$= -\frac{2}{10}P + 380$

$\Rightarrow$ Maximize

$R(P) = P\cdot Q(P)$

$= P\left(-\frac{2}{10}P + 380\right)$

$= -\frac{2}{10}P^2 + 380P$

when $P \in [0,\infty)$

$\Rightarrow R'(P) = -\frac{4}{10}P + 380 = 0$

when $P = 10\cdot\frac{380}{4} = 950$

$R''(P) = -\frac{4}{10} < 0$

$\Rightarrow R$ has max at $P = 950$

$\Rightarrow 950\$$ maximize $R(P)$

And $R(950) = -\frac{2}{10}(950)^2 + 380\cdot 950$





**Example:** From a 24 inch by 6 inch piece of cardboard, square corners are cut out so the sides fold up to form a box without a top. What should the length and width of each square be to maximize the volume of the box?

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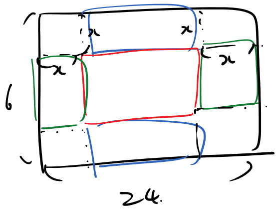


$$\text{area} = (24-2x)(6-2x) + \boxed{(6-2x)\cdot 2} + \boxed{(24-2x)\cdot 2}$$

**PRACTICE PROBLEMS**

1. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

2. A company sells $x$ mechanical pencils per year at $\$p$ per pencil. The price-demand equation for these pencils is $p = 10 - 0.001x$. What price should the company charge for the pencils to maximize revenue? What is the maximum revenue?

3. Find two (positive) numbers whose sum is 21 and product is a maximum.

4. Joe needs to build a fence to enclose a rectangular area of 800 square feet. The fence along three sides costs \$6 per foot. The material for the fourth side costs \$18 per foot (Joe needs a more expensive material for this side). Find the dimensions of the rectangle that will save Joe the most money.

5. Find two positive numbers $x$ and $y$ such that $xy = 9$ and $x + 4y$ is a minimum.

6. When a human resources company prices its training seminar at \$395 per person, 1,010 people will attend. For each \$5 increase in price, there will be 10 fewer people attending. What price should the company charge for the seminar in order to maximize its revenue?

7. Bob needs to fence in a right-angled triangular region that will border a river (see the diagram below). The fencing for the left border costs \$8 per foot, and the lower border costs \$2 per foot. Bob doesn't need any fencing along the side of the river. He has \$560 to spend, and he wants as much area as possible. What are the dimensions of the region, and what area does it enclose?