142_12c_5.3

Thursday, February 27, 2020 10:53 AM

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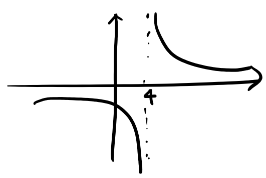
5.3 Supplement: Limits at Infinity (and Infinite Limits)

Infinite Limits

Recall from section 3.1 we concluded that a limit such as $\lim_{x\to 4} \frac{1}{x-4}$ does not exist since



$\frac{\text{number}}{0}$ is not defined.

*However, we can indicate this kind of behavior (the **way** in which this limit does not exist) by using the notation

*Thus, the notation $\lim_{x\to a} f(x) \to \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a (on either side of a) but not equal to a.

Vertical Asymptotes - The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following is true:

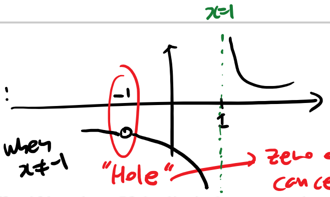
- $\lim_{x\to a} f(x) \to \infty$
- $\lim_{x\to a} f(x) \to -\infty$
- $\lim_{x\to a^-} f(x) \to \infty$
- $\lim_{x\to a^+} f(x) \to \infty$
- $\lim_{x\to a^-} f(x) \to -\infty$
- $\lim_{x\to a^+} f(x) \to -\infty$

when one of the left, or right limit is ∞ or $-\infty$

Question: How do we find and describe the behavior near vertical asymptotes? How do we find holes?

$x=1$

Graph of function:



Summary:
① f(x) has a hole at $x=-1$
② f(x) has vertical asymptote at $x=1$

$f(x) = \frac{x+3}{x-1}$ when $x \neq -1$

"Hole" → zero of canceled out equation.

Example: Find $\lim_{x\to 1} \frac{x^2+4x+3}{x^2-1}$ **algebraically**, if it exists. If the limit does not exist, use limits to describe the **way** in which it does not exist.

$$\lim_{x\to 1} \frac{x^2+4x+3}{x^2-1} = \lim_{x\to 1} \frac{(x+3)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \lim_{x\to 1} \frac{x+3}{x-1}$$

$$\lim_{x\to 1^-} \frac{x+3}{x-1} = \text{(thinking } x \text{ is close to 1 but } x<1\text{)} = \frac{+}{-} = -\infty$$

⇒ figure out sign of each term.

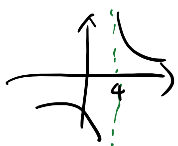
$$\lim_{x\to 1^+} \frac{x+3}{x-1} = \frac{+}{+} = +\infty$$

$\lim_{x\to 1} f(x) = \text{DNE}$

Example: Find any holes and/or vertical asymptotes of the function $f(x) = \frac{x^2+5x+6}{x^2-x-12}$ **algebraically**. If there are vertical asymptotes, use limits to describe the behavior near each asymptote.

Simplify $f(x)$. $= \frac{(x+2)\cancel{(x+3)}}{(x-4)\cancel{(x+3)}}$ ⇒ Hole: zero of $x+3$ ⇒ $x=-3$.

Vertical Asymptote: $x=4$.



$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{x+2}{x-4} = \frac{+}{-} = -\infty$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{x+2}{x-4} = \frac{+}{+} = +\infty$$

Limits at Infinity of Polynomials

$\lim_{x\to\infty} p(x) = \lim_{x\to\infty}$ "leading term"

$p(x) = a\cdot x^n + b\cdot x^{n-1} + \cdots + \text{Constant}.$

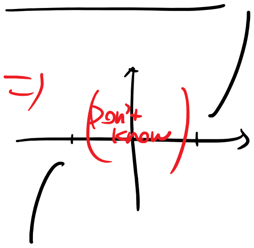
biggest degree term — "the leading term"

Example: Describe the end behavior of $p(x) = 3x^3 - 500x^2$. In other words, find $\lim_{x\to-\infty} p(x)$ and $\lim_{x\to\infty} p(x)$.

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} 3x^3 = +\infty$$

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} 3x^3 = -\infty$$

⇒



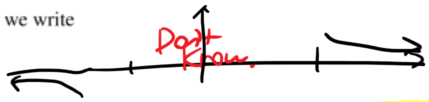
Limits at Infinity of Rational Functions

Consider the graph of $f(x) = \dfrac{1}{x-4}$ again below. As x gets larger (or smaller), we see that the values of $f(x)$ get closer to 0.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x-4} = \frac{1}{\text{growing to } \infty} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x-4} = \frac{1}{\text{growing to } -\infty} = 0$$

*Symbolically, we write

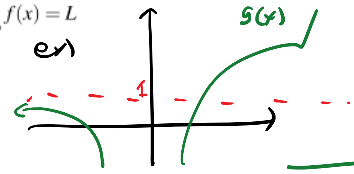


end behavior of $\frac{1}{x-4}$.

Horizontal Asymptotes - The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

ex)

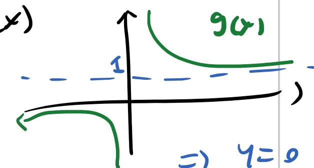


y = 1 is Horizontal Asymptote of g(x).

Question: How do we find horizontal asymptotes?

Find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

ex)



$\lim_{x\to\infty} g(x) = 1$

$\lim_{x\to-\infty} g(x) = 0$

$\Rightarrow$ $y = 0$ and $y = 1$ are Horizontal Asymptote

Calculating Limits at Infinity of Rational Functions

*If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Thus, to calculate limits at infinity (i.e. find horizontal asymptotes), we will...

Example: Find the horizontal asymptotes, **algebraically**, of the curve $y = \dfrac{2x^2+x-1}{x^4+x-2}$, if they exist.

$$\lim_{x\to\infty} \frac{2x^2+x-1}{x^4+x-2}$$

$\to$ leading term: $2x^2$

$\to$ leading term x^4

$4 > 2 \Rightarrow$ divide both top / bottom by x^4

$$= \lim_{x\to\infty} \frac{\frac{2x^2}{x^4} + \frac{x}{x^4} - \frac{1}{x^4}}{\frac{x^4}{x^4} + \frac{x}{x^4} - \frac{2}{x^4}} = \lim_{x\to\infty} \frac{\left(\frac{2}{x^2} + \frac{1}{x^3} - \frac{1}{x^4}\right) \to 0}{\left(1 + \frac{1}{x^3} - \frac{2}{x^4}\right) \to 1} = \frac{0}{1} = 0$$

$$\lim_{x\to-\infty} \frac{2x^2+x-1}{x^4+x-2} = \text{do the same thing} = \lim_{x\to-\infty} \frac{\frac{2}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}}{1+\frac{1}{x^3}-\frac{2}{x^4}} = \frac{0}{1} = 0$$

$x^{-3} + x^{-4} - 2 \cdot x^{-5}$

Example: Find the horizontal asymptotes, **algebraically**, of the curve $y = \frac{2x^5 + x - 1}{x^2 + x - 2}$, if they exist.

$$\lim_{x\to\infty} \frac{2x^5 + x - 1}{x^2 + x - 2}$$ leading term: x^5

$$= \lim_{x\to\infty} \frac{\frac{2x^5}{x^5} + \frac{x}{x^5} - \frac{1}{x^5}}{\frac{x^2}{x^5} + \frac{x}{x^5} - \frac{2}{x^5}} = \lim_{x\to\infty} \frac{\left(2 + \frac{1}{x^4} - \frac{1}{x^5}\right)^{2}}{\left(\frac{1}{x^3} + \frac{1}{x^4} - \frac{2}{x^5}\right)} = \frac{2}{0} = \frac{+}{+} = +\infty$$

$x^{-3} + x^{-4} - 2 \cdot x^{-5}$

$$\lim_{x\to-\infty} \left(\quad '' \quad \right) = \lim_{x\to-\infty} \left(\quad '' \quad \right) = \frac{2}{0} = \frac{+}{-} = -\infty$$

Example: Find the horizontal asymptotes, **algebraically**, of the curve $y = \frac{2x^2 + x - 1}{3x^2 + x - 2}$, if they exist.

ex) $f(x) = \frac{1}{x^{-4} + \text{lowest degree term}}$

$\Rightarrow \lim_{x\to-\infty} f(x) = +\infty$

* Exponential Functions in rational equation.

e^x dominate polynomial!

ex) $f(x) = e^x - x^{1001} + x^{1000} + \cdots$

leading term.

$$\lim_{x\to\infty} f(x) = +\infty$$

$-7 \cdot -\infty = +\infty$

$e^{-7 \cdot x} \to e^{+\infty} = \infty$

$$\lim_{x\to-\infty} \frac{1}{5 + e^{-7x}} = \lim_{x\to-\infty} \frac{\frac{1}{e^{-7x}} \to 0}{\frac{5}{e^{-7x}} \to 0 + 1} = \frac{0}{1} = 0.$$

leading term.

$$\lim_{x\to\infty} \frac{1}{5 + e^{-7x}} = \frac{1}{5 + 0} = \frac{1}{5}$$

$x \to \infty \Rightarrow -7x \to -\infty \Rightarrow e^{-7x} \to e^{-\infty} = 0$