142_12c_5.2

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5.2 Supplement: The Second Derivative

Second Derivative - For $y = f(x)$, the **second derivative** of f, provided that it exists, is

$$f''(x) = \frac{d}{dx} f'(x)$$

or $\frac{d^3}{dx^3}$

Other notations for $f''(x)$ are

$\frac{d}{dx} \cdot \frac{d}{dx} = \frac{d^2y}{dx^2}$. y''

y''': taking derivative three times

Example: Find y'' for $y = 8^x \ln x$. Do not simplify your answer.

$$y' = (8^x)' \cdot \ln x + 8^x \cdot (\ln x)'$$

$$= (\ln 8) \cdot 8^x \cdot \ln x + 8^x \cdot \frac{1}{x} = \ln 8\, y + 8^x \cdot \frac{1}{x}$$

$$y'' = \left(\ln 8\, y + 8^x \frac{1}{x}\right)' = \ln 8 \cdot y' + \left(8^x \cdot \frac{1}{x}\right)'$$

$$= \ln 8\left(\ln 8 \cdot 8^x \cdot \ln x + 8^x \frac{1}{x}\right) + (8^x)' \cdot \frac{1}{x} + 8^x \cdot \left(\frac{1}{x}\right)'$$

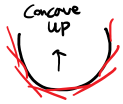
$$= \qquad + \ln 8 \cdot 8^x \cdot \frac{1}{x} + 8^x \cdot \left(-\frac{1}{x^2}\right)$$

$(x^{-1})' = -x^{-2} = -\frac{1}{x^2}$

Concavity Test

- The graph of a function f is concave upward on the interval (a,b) if $f'(x)$ is increasing on (a,b). In other words, if $f''(x) > 0$. Why? First derivative test on $f'(x)$.

Concave up



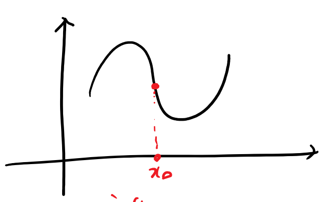
— : tangent line

- The graph of a function f is concave downward on the interval (a,b) if $f'(x)$ is decreasing on (a,b). In other words, if $f''(x) < 0$. Why? First derivative test on $f'(x)$

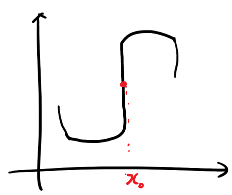
Concave down.

- An inflection point is a point on the graph where the concavity changes! (at x) $\Longleftrightarrow$ $f''(x) = 0$ or DNE and concavity changes $+ \to -$ or $- \to +$

Some examples include...



inflection pt



$f''(x_0)$ DNE. at x_0

Question: What do we notice?

Answer: If $y = f(x)$ has an inflection point at $x = c$, then either $f''(c) = 0$ or $f''(c)$ does not exist, f(x) is defined, and $f''(x)$ changes sign at $x = c$.

*In order to determine where a function is concave up/down and has inflection points, we will create a sign chart for $f''(x)$. What "important" x values will we put on our chart?

Example: Find where the function $f(x) = \ln(x^2 - 4x + 5)$ is concave upward/downward, and find any inflection points.

① find $f''(x)$ $y = x^2 - 4x + 5$ $\frac{dy}{dx} = 2x - 4$

$f(x) = \ln(y)$

$$\frac{df}{dx} = (\ln(y))' \cdot \frac{dy}{dx} = \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{y}(2x-4) = \frac{2x-4}{x^2-4x+5}$$

$$f''(x) = \frac{(x^2-4x+5)\cdot 2 - (2x-4)^2}{(x^2-4x+5)^2} = \frac{2x^2-8x+10-(4x^2-16x+16)}{\sim} = \frac{-2x^2+8x-6}{\sim} = \frac{-2(x^2-4x+3)}{\sim}$$

Quotient rule, Expand

$$= \frac{-2(x-1)(x-3)}{(x^2-4x+5)^2} \quad (= f''(x))$$

Split up. ⇒ Bottom is positive

② when $f''(x) = 0$ or DNE.

1) DNE :⇒ Bottom $(x^2 - 4x + 5)^2 = 0$ → (half of this coefficient)^2

$x^2 - 4x + 4 + 1$

$= (x-2)^2 + 1$

Since $(x-2)^2 \geq 0$

⇒ $(x-2)^2 + 1 > 0$

⇒ No x make Bottom 0!

2) $f''(x) = 0$:⇒ Top part should be 0.

So Top = 0 if $x = 1, 3$.

⇒ $f''(x) = 0$ or DNE when $x = 1, 3$.

③ Make Sign chart

Sign of $f''(x)$

x_0	1	x_0	3	x_0
$-$		$+$		$-$
$-2(x-1)(x-3)$: $-$ $-$		$-2(x-1)(x-3)$: $+$ $-$		$-2(x-1)(x-3)$: $+$ $+$
$- \cdot - \cdot - = -$		$- \cdot + \cdot - = +$		$- \cdot + + = -$
$f''(x) < 0$		$f''(x) > 0$		$f''(x) < 0$

④ Conclusion

f is concave down when $x \in (-\infty, 1) \cup (3, \infty)$

f is " up when $x \in (1, 3)$

Inflection pts occur at $x = 1$, $x = 3$

(i.e. $(1, f(1))$, $(3, f(3))$ are the inflection points of f)

Example: Find where the function $f(x) = 8e^x - e^{2x}$ is concave upward/downward, and find any inflection points.

① $f''(x)$ $f'(x) = 8e^x - 2e^{2x}$

$f''(x) = 8e^x - 2\cdot(2e^{2x})$

$= 8e^x - 4e^{2x}$

$(e^{2x})' = e^{2x} \cdot (2x)'$

$= 2 \cdot e^{2x}$

$(e^{f(x)})' = f'(x) \cdot e^{f(x)}$

$e^{\ln t} = t$

$\ln e^t = t$

② $f''(x) = 0$ or DNE.

1) $f''(x) = 4e^x(2 - e^x)$ → 0 at $x = \ln 2$ ⇒ $f''(x) = 0$ at $x = \ln 2$

Cannot be 0

2) $f''(x)$ DNE. Since e^x is defined everywhere, No such point!

⇒ $f''(x) = 0$ or DNE happens when $x = \ln 2$.

③ Sign Chart!

x	$\ln 2$	x
$f''(x) = 4e^x(2 - e^x)$: $+$ $+$		$f''(x) = 4e^x(2 - e^x)$: $+$ $-$
$f''(x) > 0$		$f''(x) < 0$

④ Conclusion.

f is concave down if $x \in (\ln 2, \infty)$

" up if $x \in (-\infty, \ln 2)$

f has inflection pt at $(\ln 2, f(\ln 2))$

Point of Diminishing Returns

*If a company decides to increase spending on advertising, for example, it would expect sales to increase. At first, sales will increase at an increasing rate and then increase at a decreasing rate. The value of x where the **rate of change** of sales changes from **increasing** to **decreasing** is called the **point of diminishing returns**.

*This is also the point where the **rate of change** has a ________________ value. Money spent after this point may increase sales, but at a lower rate.

inflection pt from $+ \to -$

In other words....

Example: An appliance store is selling 200 ovens monthly. If the store invests \$$x$ thousand in an advertising campain, the ad company estimates that sales will increase to

$$N(x) = 3x^3 - 0.25x^4 + 200$$

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where $0 \le x \le 9$.

When is the rate of change of sales with respect to advertising increasing and when is it decreasing? What is the point of diminishing returns and the maximum rate of change of sales? Sketch a graph of $N(x)$ and $N'(x)$ and locate the point of diminishing returns.

The Second Derivative Test (for Local/Relative Extrema)

Suppose that f is defined on (a,b) and c is a critical value of f such that $f'(c) = 0$.

- If $f''(c) > 0$, then $f(c)$ is a local/relative minimum.
- If $f''(c) < 0$, then $f(c)$ is a local/relative maximum.

If ① c is critical value of f
② f(c) is defined.

NOTE: If $f''(c) = 0$ or $f''(c)$ does not exist, then the Second Derivative Test fails, and we have to use some other method to find the local extrema (i.e., we have to use the First Derivative Test we discussed in the previous section).

NOTE: We can only *attempt* to use the Second Derivative Test for critical values of f such that $f'(c) = 0$. If f has a critical value such that $f'(c)$ does not exist, then we cannot use the Second Derivative Test to find the local extrema (we have to use the First Derivative Test).

Example: Use the Second Derivative Test to find any local extrema of the function $f(x) = 1 + 9x + 3x^2 - x^3$, if possible. If it is not possible, explain why.

$f'(x) = 9 + 6x - 3x^2$

$f''(x) = 6 - 6x$

① $f''(x) = 6 - 6x$

② Critical value of f.

$\Rightarrow f'(x) = 0$ or DNE. → cannot happen since $f'(x)$ is polynomial.

$f'(x) = -3x^2 + 6x + 9$

$= -3(x^2 - 2x - 3) = -3(x+1)(x-3)$

$f'(x) = 0$ when $x = -1, 3$

③ Check $f''(c)$ for c: critical values.

$f''(-1) = 6 - 6(-1) = 6 + 6 = 12 > 0$

$f''(3) = 6 - 6(3) = 6 - 18 = -12 < 0$

④ Conclusion

$f''(-1) > 0 \Rightarrow (-1, f(-1))$ is local min.

$f''(3) < 0 \Rightarrow (3, f(3))$ is " max.