142_12c_4.3_4.4

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4.3 and 4.4 Supplement: The Chain Rule and Derivatives of Exponential and Log Functions

$\boxed{y=f(u)}$ u is variable

$\boxed{u=g(x)}$

If $y = f(u)$ and $u = g(x)$, then we can express y as a function of x as follows:

$$y = f(u) = f[g(x)] = m(x)$$

$y = f \circ g(x) = f(g(x))$

where $m(x)$ is called the **composite** of the two functions f and g.

*The domain of m is the set of all numbers x such that x is in the domain of g and $g(x)$ is in the domain of f.

2nd expression

$$\boxed{\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}}$$

1st expression

$$\boxed{y' = f'(g(x)) \cdot g'(x)}$$

Chain Rule

If $y = f(u)$ and $u = g(x)$, then define the composite function

$$y = m(x) = f[g(x)]$$

then "form 1" of the chain rule is

$$m'(x) = f'[g(x)]\,g'(x)$$

(provided that $f'[g(x)]$ and $g'(x)$ exist), or, equivalently, "form 2" of the chain rule is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

(provided that dy/du and du/dx exist).

① find $f'(u)$

② put $u = g(x)$ on $f'(u)$

③ find $g'(x)$,

④ $y' = f'(g(x)) \cdot g'(x)$

Some General Chain Rule Formulas (Use with Form 1)

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$$

$y = u^n$ $\quad y = (f(x))^n$

$u = f(x)$

$y' = \Big($ ① differentiation of u^n ② put $f(x)$ inside of the differentiation $\cdot f'(x) \Big) = n(f(x))^{n-1} \cdot f'(x)$

①: $\frac{d}{du}(u^n) = nu^{n-1}$

② $(u^n)' = n(f(x))^{n-1}$ with $u = f(x)$

③ $f'(x)$

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} f'(x)$$

$y = e^u \rightarrow \frac{dy}{du} = e^u$

$u = f(x)$ $\quad \frac{du}{dx} = f'(x)$

$y' = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot f'(x)$

$\boxed{y' = e^{f(x)} \cdot f'(x)}$

$$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)} f'(x)$$

$y = \ln u$ $\quad \frac{dy}{du} = \frac{1}{u}$

$u = f(x)$ $\quad \frac{du}{dx} = f'(x)$

$y' = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot f'(x)$

$\boxed{y' = \frac{1}{f(x)} \cdot f'(x)}$

$$\frac{d}{dx}b^{f(x)} = b^{f(x)}(\ln b) f'(x)$$

$y = b^u$ $\quad \frac{dy}{du} = \ln b \cdot b^u$

$u = f(x)$ $\quad \frac{du}{dx} = f'(x)$

$y' = \frac{dy}{du} \cdot \frac{du}{dx} = \ln b\, b^u f'(x)$

$\boxed{= b^{f(x)} \cdot \ln b\, f'(x)}$

$$\frac{d}{dx}\log_b[f(x)] = \frac{1}{\ln b}\left(\frac{1}{f(x)}\right) f'(x)$$

$y = \log_b u$ $\quad \frac{dy}{du} = \frac{1}{\ln b} \cdot \frac{1}{u}$

$u = f(x)$ $\quad \frac{du}{dx} = f'(x)$

$y' = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\ln b} \cdot \frac{1}{u} \cdot f'(x)$

$\boxed{= \frac{1}{\ln b} \cdot \frac{1}{f(x)} \cdot f'(x)}$

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Example: Find the derivative of each of the following functions. (Do not simplify your answers.)

a) $f(x) = \sqrt[5]{x^3 + 4x^2 - 7}$

$f(x) = y = \sqrt[5]{x^3+4x^2-7}$

$\boxed{\text{Set: } u = x^3 + 4x^2 - 7}$

diff: $3x^2$ $8x$ 0

$y = \sqrt[5]{u} = u^{\frac{1}{5}} \Rightarrow \frac{dy}{du} = \frac{1}{5} \cdot u^{-\frac{4}{5}}$

$\frac{du}{dx} = 3x^2 + 8x$

$y' = f'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{1}{5} \cdot u^{-\frac{4}{5}} \cdot (3x^2 + 8x)$

$\boxed{f'(x) = \frac{1}{5} \cdot (x^3 + 4x^2 - 7)^{-\frac{4}{5}} \cdot (3x^2+8x)}$

b) $f(x) = e^{x^2+5x}$

"$y = e^{x^2+5x} = e^u$ $\qquad \frac{dy}{du} = e^u$

$u = x^2 + 5x$ $\qquad \frac{du}{dx} = 2x+5$

$f'(x) = y' = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (2x+5) = e^{x^2+5x} \cdot (2x+5)$

$\boxed{f'(x) = e^{x^2+5x} \cdot (2x+5)}$

$y = f(g(h(x)))$

$y' = f'(g(h(x))) \cdot (g(h(x)))'$

Chain rule.

$= \quad " \quad g'(h(x)) \cdot h'(x)$

$" = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$

c) $f(x) = \log_3(5^{x^2-1})$

$u = 5^{x^2-1}$

$y = \log_3 u$

$\frac{dy}{du} = \frac{1}{\ln 3} \cdot \frac{1}{u}$

$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = (\ln 5) \cdot 5^v \cdot 2x = (\ln 5) \cdot 5^{x^2-1} \cdot (2x)$

$v = x^2 - 1$

$u = 5^v$

$\frac{du}{dv} = (\ln 5) \cdot 5^v$ → exponent rule

$\frac{dv}{dx} = 2x$

$\frac{dv}{dx} = v'$

$y' = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{1}{\ln 3} \cdot \frac{1}{u}\right) (\ln 5) \cdot 5^{x^2-1} \cdot (2x)$

$= \frac{1}{\ln 3} \cdot \frac{1}{5^{x^2-1}} \cdot \ln 5 \cdot 5^{x^2-1} (2x)$

HW: Calculate $f'(x)$ using chain rule only once.

d) $f(x) = \frac{1}{\ln|1-x^3|}$

$u = 1 - x^3$

$y = \frac{1}{\ln|u|}$

$\frac{dy}{du} = \frac{dy}{dv} \cdot \frac{dv}{du} = -\frac{1}{v^2} \cdot \frac{1}{|u|} = -\frac{1}{(\ln|u|)^2} \cdot \frac{1}{|u|}$

$y = \frac{1}{v} = v^{-1}$ $\qquad \frac{dy}{dv} = -v^{-2} = -\frac{1}{v^2}$

$v = \ln|u|$

$\frac{dv}{du} = \frac{1}{|u|}$

$\frac{du}{dx} = -3x^2$

$y' = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{(\ln|u|)^2} \cdot \frac{1}{|u|} \cdot (-3x^2) = -\frac{1}{(\ln|1-x^3|)^2} \cdot \frac{1}{|1-x^3|} (-3x^2)$

e) $f(x) = e^{5x}\sqrt{2x^3 - 4^x + 7e^{4x-9}}$

$= 5 \cdot e^{5x} \cdot \sqrt{2x^3 - 4^x + 7e^{4x-9}} + e^{5x} \cdot (\sqrt{\sim})'$

prod rule

$(\underbrace{\sqrt{2x^3 - 4^x + 7e^{4x-9}}}_{=y})' = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} \cdot u^{-\frac{1}{2}} \cdot (6x - (\ln 4) \cdot 4^x + 28e^{4x-9})$

$u = \boxed{2x^3 - 4^x + 7e^{4x-9}} \longrightarrow \frac{du}{dx} = 6x - (\ln 4)4^x + (7e^{4x-9})'$

$(7e^{4x-9})' = 28 \cdot e^{4x-9}$

$y = \sqrt{u} = u^{\frac{1}{2}} \longrightarrow \frac{dy}{du} = \frac{1}{2} \cdot u^{-\frac{1}{2}}$

$a = 7e^{4x-9} = 7e^v \qquad \frac{da}{dv} = 7e^v$

$v = 4x - 9 \qquad \frac{dv}{dx} = 4$

$a' = (7e^{4x-9})' = \frac{da}{dv} \cdot \frac{dv}{dx} = 7e^v \cdot 4 = 28e^v$

$= 28 \cdot e^{4x-9}$

f) $f(x) = \sqrt[5]{\left(\frac{x^4 - 5x + 3}{2x^2 + \pi x - 4}\right)^3}$

$y =$

Exercise!

$u = \left(\frac{x^4 - 5x + 3}{2x^2 + \pi x - 4}\right) \longrightarrow y = u^{\frac{3}{5}}$

$\frac{du}{dx} =$ the quotient rule. $\qquad y' = \frac{dy}{du} \cdot \frac{du}{dx} = \sim$

$\frac{dy}{du} = \frac{3}{5} \cdot u^{-\frac{2}{5}}$

Showing Form 1 is Equivalent to Form 2

$\boxed{y'}$ $\qquad \frac{dy}{dx}$

Example: Let $y = \ln u$ and $u = 2x^6 + x^2$. Find dy/dx using Form 2 of the chain rule. Then, use Form 1 to compute the derivate and compare your answers.

y'

Example: If $f(t) = t^2 + 3$ and $t(w) = 6\ln w$, find df/dw using Form 2 of the chain rule.

If you change of production of the item when you already made 10 items, then prize will decrease by $2600e^{-10}$ \$.

Example: The price-demand function of a company that makes x items of a certain commodity each week is given by $p(x) = 2600e^{-x}$, where p is the price in dollars of each item.

a) Find $p'(10)$ and interpret your answer.

$u = -x$ $\quad \frac{du}{dx} = -1$

$P = 2600 \cdot e^u$

$\frac{dP}{du} = 2600 \cdot e^u$

$$P'(x) = \frac{dP}{du} \cdot \frac{du}{dx}$$
$$= 2600 \cdot e^u \cdot (-1)$$
$$= -2600 \cdot e^{-x}$$
$$\boxed{P'(10) = -2600 \cdot e^{-10}}$$

b) Find the marginal revenue when 10 items are sold each week and interpret your answer.

$R(x) = p(x) \cdot x$
$= 2600 \cdot x e^{-x}$

When you make 1 more item from now, then you earn $R'(10)$ \$'s for the 11th item.

Marginal Rev $R'(x) = (x)' \cdot p(x) + x \cdot p'(x)$
$= 1 \cdot p(x) + x \cdot (-2600) e^{-x}$
$= 2600 \cdot e^{-x} + x(-2600) e^{-x}$
$= 2600 e^{-x}(1 - x)$

$R'(10) = 2600 e^{-10}(1-10) = n$

Example: The total cost (in hundreds of dollars) of producing x cameras per week is $C(x) = 6 + \sqrt{4x+4}$, where $0 \leq x \leq 30$.

a) Find $C(24)$ and $C'(24)$, and then interpret both results.

b) Estimate total cost when 25 cameras are produced.

c) Find the exact cost when 25 cameras are produced.

d) Approximate the cost of the 16th camera.

e) Find the exact cost of the 16th camera.