Derivatives of Exponential and Logarithmic Functions

1. $\frac{d}{dx}e^x = e^x$
2. $\frac{d}{dx}\ln x = \frac{1}{x}$
3. $\frac{d}{dx}b^x = b^x \ln b$
4. $\frac{d}{dx}\log_b x = \frac{1}{\ln b}\left(\frac{1}{x}\right)$

$\boxed{\log_b x = \frac{\ln x}{\ln b}}$ $\boxed{b = e^{\ln b}} = b^{\ln e}$ $\ln e = 1$

$b^x = (e^{\ln b})^x = (e^{x\ln b})$

$(b^x)' = \ln b \cdot e^{x\ln b} = \ln b \cdot b^x$

$\frac{d}{dx}\log_b x = \frac{d}{dx}\frac{\ln x}{\ln b}$

$= \frac{1}{\ln b}\frac{d}{dx}\ln x$ constant

$= \frac{1}{\ln b}\cdot\frac{1}{x}$

Example: Find $\frac{dy}{dx}$ for the following. **Do not simplify your answer!**

a) $y = \log_\pi x$ $\quad \frac{dy}{dx} = \left(\begin{matrix} b=\pi \\ \text{Rule } 4\end{matrix}\right)$ $\quad \frac{dy}{dx} = \frac{1}{\ln\pi}\cdot\frac{1}{x}$

b) $y = 7.129^x$ $\quad \frac{dy}{dx} = \left(\begin{matrix}\text{Rule } 3 \\ b=7.129\end{matrix}\right) = (7.129)^x \ln(7.129)$

c) $y = 8e^x - e^8 + \frac{1}{9}\ln x$ $\quad \frac{dy}{dx} = 8e^x + 0 + \frac{1}{9}\cdot\frac{1}{x}$

d) $y = x^4 - 4^x - 3\log_6 x + 7(5^x)$ $\quad b=6$

$\frac{dy}{dx} = 4x^3 - 4^x \ln 4 - 3\cdot\frac{1}{\ln 6}\cdot\frac{1}{x} + 7\cdot 5^x \cdot \ln 5$

e) $y = \frac{x^3 - 3x(2^x) - \frac{4}{11}}{x}$

$= x^2 - 3(2^x) - \frac{4}{11}x^{-1}$

$\frac{dy}{dx} = 2x - 3(2^x \cdot \ln 2) - \frac{4}{11}\cdot(-x^{-2})$

$= 2x - (3\ln 2)\cdot 2^x + \frac{4}{11}x^{-2}.$

Note: In some cases it might be necessary to simplify a function using logarithmic properties before taking the derivative.

Example: Find $f'(x)$ if $f(x) = 5 + 7\ln\frac{6}{x^3}$.

$\ln\frac{6}{x^3} = \ln 6 - \ln x^3$

$= \ln 6 - 3\ln x$

$= 5 + 7(\ln 6 - 3\ln x)$

$f(x) = 5 + 7\ln 6 - 21\ln x$

$f'(x) = 0 \quad 0 \quad -21\cdot\frac{1}{x} \quad = -21\cdot\frac{1}{x}$

$f(e) = 1 + \ln e^4 = 1 + 4\ln e = 1 + 4 = 5$

Example: Find the equation of the line tangent to the graph of $f(x) = 1 + \ln x^4$ at $x = e$.

$f(x) = 1 + \ln x^4$
$= 1 + 4\ln x$

$f'(x) = 0 + 4 \cdot \frac{d}{dx}\ln x = \frac{4}{x}$

② It contains $(e, f(e)) = (e, 5)$

① Slope $= f'(e) = \frac{4}{e}$

③ slope point form: $y - 5 = \frac{4}{e}(x - e)$
$y = \frac{4}{e}x + 5 - 4 = \frac{4}{e}x + 1$

Example: The price-demand equation of a store that sells x hats at a price of p dollars per hat is given by $p = 350(0.999)^x$. Find the rate of change of price with respect to demand when the demand is 800 hats. Then, interpret your result.

p'

$p(x) = 350 \cdot (0.999)^x$ $b = 0.999$ in the rule

$p'(x) = 350 \cdot \frac{d}{dx}\left((0.999)^x\right) = 350 \cdot \left((0.999)^x \cdot \ln(0.999)\right)$

$p'(800) =$ rate of change. (Use calculator)

Result: when demand is 800 hats, then change of demand by 1 increases price by $p'(800)$

4.2 Supplement: Derivatives of Product and Quotients

The Product Rule - If $y = f(x) = F(x)S(x)$ and if $F'(x)$ and $S'(x)$ exist, then

$$f'(x) = F(x)S'(x) + S(x)F'(x)$$

Or, we could write...

Example: Find y' if $y = 2x^2(3x^4 - 5)$.

$F(x) = 2x^2 \qquad S(x) = 3x^4 - 5$

$F'(x) = 4x \qquad S'(x) = 12x^3$

$y' = F \cdot S' + S \cdot F' = (2x^2) \cdot 12x^3 + (3x^4 - 5)4x$

$= 24x^5 + 12x^5 - 20x = 36x^5 - 20x$

Example: Find $f'(x)$ if $f(x) = (x^2+3)(\sqrt[4]{x} + \sqrt[8]{x^3})$.

$= (x^2+3)(x^{\frac{1}{4}} + x^{\frac{3}{8}})$

$F(x) = x^2+3 \qquad S(x) = x^{\frac{1}{4}} + x^{\frac{3}{8}}$

$F'(x) = 2x \qquad S'(x) = \frac{1}{4}x^{-\frac{3}{4}} + \frac{3}{8} \cdot x^{-\frac{5}{8}}$

$f'(x) = (x^2+3)(\frac{1}{4} \cdot x^{-\frac{3}{4}} + \frac{3}{8} x^{-\frac{5}{8}}) + 2x(x^{\frac{1}{4}} + x^{\frac{3}{8}})$

Example: Find $f'(t)$ if $f(t) = 10^t \log t$.

$F(t) = 10^t \qquad S(t) = \log t = \log_{10} t$

$F'(t) = 10^t \cdot \ln 10 \qquad S'(t) = \frac{1}{\ln 10} \cdot \frac{1}{t}$

$f'(x) = 10^t \cdot \frac{1}{\ln 10} \cdot \frac{1}{t} + 10^t \ln 10 \cdot \log t$

$= 10^t\left(\frac{1}{\ln 10} \cdot \frac{1}{t} + \ln 10 \cdot \log t\right)$

Example: Find $f'(x)$ if $f(x) = \pi x \log x^5$. $= 5\pi x \log x$

from $(\log x^5 = 5 \log x)$

$F(x) = 5\pi x \qquad S(x) = \log x$

$F'(x) = 5\pi \qquad S'(x) = \frac{1}{\ln 10} \cdot \frac{1}{x}$

$f'(x) = \frac{5\pi \cancel{x}}{\ln \cdot 10} \cdot \frac{1}{\cancel{x}} + 5\pi \cdot \log x$

$= \left(\frac{5\pi}{\ln 10}\right) + 5\pi \log x$.

Example: Find y' if $y = \left(3w^2 + 4\log_3\left(\frac{6}{w^2}\right)\right)(2^w + 3e^w)$.

$F(x) = 3w^2 + 4\log_3\left(\frac{6}{w^2}\right)$

$= 3w^2 + 4\log_3 6 - 8\log_3 w$

$\boxed{F'(x) = 6w + \frac{1}{0} - 8 \cdot \frac{1}{\ln 3} \cdot \frac{1}{w}}$

$S(x) = 2^w + 3e^w$

$S'(x) = 2^w \cdot \ln 2 + 3 \cdot e^w$

w^{-2}

$4 \cdot \left(\log_3\left(\frac{6}{w^2}\right)\right) = 4\left(\log_3 6 + \log_3\left(\frac{1}{w^2}\right)\right) = 4\left(\log_3 6 + \log_3 w^{-2}\right)$

$= 4\left(\log_3 6 - 2\log_3(w)\right) = 4\log_3 6 - 8 \cdot \log_3 w$.

$y' = F \cdot S' + F' \cdot S = \left(3w^2 + 4\log_3\left(\frac{6}{w^2}\right)\right)(2^w \ln 2 + 3e^w)$

$+ \left(6w - \frac{8}{\ln 3} \cdot \frac{1}{w}\right)(2^w + 3e^w)$

$+\left(6w - \frac{8}{\ln 3}\cdot\frac{1}{w}\right)\left(2^w + 3e^w\right)$

The Quotient Rule - If $y = f(x) = \frac{T(x)}{B(x)}$ and if $T'(x)$ and $B'(x)$ exist, then

$$f'(x) = \frac{B(x)T'(x) - T(x)B'(x)}{[B(x)]^2}$$

similar to product rule but (−) sign.

Or, we could write...

$$= \underline{T'(x)\cdot\frac{1}{B(x)} - T(x)\cdot\frac{B'(x)}{B(x)^2}}$$

$\left(\left(\frac{1}{B(x)}\right)' = -\frac{B'(x)}{B(x)^2}\right)$

Example: Find $f'(x)$ if $f(x) = \frac{2\sqrt{x}}{x^2 - 3x + 1}$. $= \frac{2x^{\frac{1}{2}}}{x^2-3x+1}$

$T(x) = 2x^{\frac{1}{2}}$ $\quad B(x) = x^2 - 3x + 1$

$T'(x) = 1\cdot x^{-\frac{1}{2}}$ $\quad B'(x) = 2x - 3$

$$f'(x) = \frac{(x^2-3x+1)\,x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}\cdot(2x-3)}{(x^2-3x+1)^2}$$

$=$

Example: $\frac{d}{du}\frac{4u^2e^u}{\log_7 u + 5\ln u}$. $= A(u)$

$T(u) = 4u^2 e^u$ $\quad B(u) = \log_7 u + 5\ln u$

$F(u) = 4u^2$ $\quad S(u) = e^u$ $\quad B'(u) = \frac{1}{\ln 7}\cdot\frac{1}{u} + \frac{5}{u}$

$F'(u) = 8u$ $\quad S'(u) = e^u$

product rule: $T'(u) = 4u^2\cdot e^u + 8u\cdot e^u = (4u^2 + 8u)e^u$

$$\frac{d}{du}A(u) = \frac{\left((\log_7 u + 5\ln u)(4u^2 + 8u)e^u - 4u^2e^u\left(\frac{1}{\ln 7}\cdot\frac{1}{u} + \frac{5}{u}\right)\right)}{(\log_7 u + 5\ln u)^2}$$

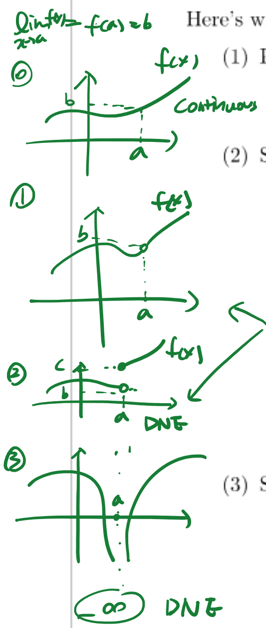
Example; Find y' if $y = \frac{2x-1}{(x^3+2)(x^2-3)}$.

Solution.

Example: Find $\frac{dy}{dx}$ if $y = \frac{x^5 - 3x + 1}{23\sqrt[4]{x}}$.

Math 142 Fall 2019 - Weak 3 - Review-Midterm
2/5/2020 -

Here's what you need to know to get the perfect grade.



(1) Precalculus

① $\sqrt[a]{x^b} = x^{\frac{b}{a}}$ ③ $\log x^u = u \cdot \log x$

- (a) Simplify the given equation. ② $\log a b = \log a + \log b$
- (b) Properties of logarithm and exponential functions. ②, ③

(2) Section 3.1 dom: e^x : $\log x \Rightarrow x > 0$

- (a) How to find the limit of function algebraically? 1) Rational: $\lim_{x \to a} \frac{f(x)}{g(x)}$ ① plug in a first 1-1: If $g(a) \neq 0 \Rightarrow$ limit $= \frac{f(a)}{g(a)}$ ② Otherwise, simplify $\frac{f(x)}{g(x)}$
  - (i) Case 1: Function consisting of rational, root.
  - (ii) Case 2: Piecewise function. Be careful about boundary!
- (b) How to find the limit of function numerically using the calculator? ① Put the number approach to the limit ② Calculate!
- (c) How to find the limit of function from the graph?
- (d) Can you figure out the points where limit does not exists from the graph of a function?
- (e) What is the definition of continuity? For given a point x where $f(x)$ is not continuous, can you figure out which conditions of the continuity fail?
- (f) Can you find an interval where the given function is continuous? (= find domain)

(3) Section 3.2

- (a) How to find the slope of a secant line over two points? $(a, f(a))$ $(b, f(b)) \Rightarrow$ slope: $\frac{f(a)-f(b)}{a-b}$ $y - f(a) = \left(\frac{f(a)-f(b)}{a-b}\right) \cdot (x-a)$ secant line containing $(a, f(a)), (b, f(b))$
- (b) How to find the slope of a tangent line over two points? Can you figure out this graphically?
- (c) What is difference between "instantaneous rate of change" and "average rate of change"? tangent secant
- (d) Can you interpret your calculations from the real world situation?

$a = b + h$
$f(a) = f(b+h)$
Slope of tangent line: $\lim_{h \to 0} \frac{f(b+h) - f(b)}{\cancel{b}+h-\cancel{b}} =$

(4) Section 3.3

- (a) How to find the derivative using the definition of derivative? Slope of tan line
- (b) Can you sketch $f(x)$ from $f'(x)$ or vice versa precisely?
- (c) How to find an equation of tangent line using the derivative. Specify $f'(x) = 0$ and inflection pt.

(5) Section 4.1 Remember: Rules of derivative

- (a) Can you find the derivative using rules?
- (b) Can you find the derivative of the function containing exponential and logarithmic terms?
- (c) Do you know the meaning of "total cost" and "marginal cost"?
- (d) Can you approximate the revenue using the derivative?

(6) Section 4.2

- (a) Can you find the derivative of the multiplication of two functions?
- (b) Can you find the derivative of the rational function?