142_12c_4.1

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4.1 Supplement: Derivatives of Powers, Exponents, and Sums

Derivative Notation - If $y = f(x)$, then

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{d}{dx}f(x)$$

all represent the derivative of f at x.

$f(x)$

$f'(x) = $ (limit (when $h \to 0$) difference quotient between $(x+h, f(x+h))$ $(x, f(x))$)

$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{x+h-x}$

$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

= Slope of tangent line at x.

Derivative Rules:

1) If $f(x) = c$, where c is a constant, then $f'(x) = 0$. **(Constant Function Rule)**

constant

2) If $f(x) = ax + b$, then $f'(x) = a$. **(Derivative of a Linear Function)**

3) If $f(x) = x^n$, where n is any nonzero real number, then $f'(x) = nx^{n-1}$. **(Power Rule)**

$y = t^{-3}$ $\quad y' = -3 \cdot t^{-3-1} = -3 \cdot t^{-4}$

4) If $f(x) = ku(x)$, where k is a constant, then $f'(x) = ku'(x)$. **(Constant Multiple Rule)**

5) If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$. If $f(x) = u(x) - v(x)$, then $f'(x) = u'(x) - v'(x)$. **(Sum and Difference Rules)**

Example: $y = \sqrt{7}$. Find y'. $= 0$ since it is constant.

① function $f(x)$ or y

Example: $f(x) = x^5$. Find $f'(x)$. $= 5x^4$

(Newton) ② Derivative 1st notation.

$f'(x)$ or y'

⇒ Derivative of $f(x)$ 〃 y

Example: $y = t^{-3}$. Find $\frac{dy}{dt}$. $= -3 \cdot t^{-3-1} = -3t^{-4}$

variable = t

(Leibniz) ③ Derivative 2nd notation

$\frac{df(x)}{dx}$ or $\frac{dy}{dx}$

$\frac{d}{dx} \cdot f(x)$

Example: $\frac{d}{dx}\frac{1}{\sqrt[3]{x^2}}$. $= \frac{d\left(\frac{1}{\sqrt[3]{x^2}}\right)}{dx} = -\frac{2}{3} \cdot x^{-\frac{5}{3}}$

$\frac{d}{dx}\frac{1}{(x^2)^{\frac{1}{3}}} = \frac{d}{dx}\left(\frac{1}{x^{\frac{2}{3}}}\right) = \boxed{\frac{d}{dx}x^{-\frac{2}{3}}} = -\frac{2}{3}x^{-\frac{2}{3}-1}$

$= -\frac{2}{3}x^{-\frac{5}{3}}$

ex) $\frac{2}{3} = \frac{1}{3} \cdot 2$

Example: $f(x) = 3x^2$. Find $f'(x)$. $= 3 \cdot$ (derivative of x^2) $= 3 \cdot 2x = 6x$.

$x^2 \longrightarrow 2x$

Example: $y = 2x$. Find y'. $= 2 \cdot (\text{derivative of } x) = 2 \cdot 1 = 2$.

$2 \quad x \mapsto 1$

Example: $f(x) = 3x^2 + 7x - 9$. Find $f'(x)$.

$f'(x) = 6x + 7 - 0 = 6x + 7$.

Example: $y = \sqrt[3]{w} - 3w$. Find $\frac{dy}{dw}$.

① Change terms with exponent form

② Use power rule to each term

$y = w^{\frac{1}{3}} - 3w$

$\frac{dy}{dw} = \frac{1}{3} w^{-\frac{2}{3}} - 3$

Example: $\frac{d}{dx} \frac{3x^2 + x^4}{5\sqrt{x}} = \frac{d}{dx}\left(\frac{3x^2}{5\sqrt{x}} + \frac{x^4}{5\sqrt{x}}\right)$

$// \ \frac{3}{5} x^2 \cdot \frac{1}{\sqrt{x}} = \frac{3}{5} \cdot x^2 \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{3}{5} \cdot x^2 \cdot x^{-\frac{1}{2}}$

$= \frac{3}{5} \cdot x^{2-\frac{1}{2}}$

$x^{\frac{1}{2}}$

$= \frac{d}{dx}\left(\frac{3}{5} \cdot x^{2-\frac{1}{2}} + \frac{1}{5} x^{4-\frac{1}{2}}\right)$ ①

$= \frac{3}{5} \cdot (2 - \frac{1}{2}) \cdot x^{2-\frac{1}{2}-1} + \frac{1}{5} (4 - \frac{1}{2}) \cdot x^{4-\frac{1}{2}-1}$ ②

$= \frac{3}{5} \cdot \frac{3}{2} \cdot x^{\frac{1}{2}} + \frac{1}{5} \cdot \left(\frac{7}{2}\right) \cdot x^{\frac{5}{2}}$

$= \frac{9}{10} x^{\frac{1}{2}} + \frac{7}{10} x^{\frac{5}{2}}$ Simplification.

Applications

Example: An object moves along the y axis (marked in feet) so that its position at time t (in seconds) is $s(t) = t^3 - 6t^2 + 9t$. Find

derivative of s

$s(t) = t^3 - 6t^2 + 9t$

a) The instantaneous velocity function v.

$v = s'(t) = 3t^2 - 12t + 9$

b) The velocity at $t = 2$ and $t = 5$ seconds.

$v(5) = 3\cdot 25 - 12\cdot 5 + 9 = 75 - 60 + 9 = 24$

$v(2) = 3\cdot 4 - 12\cdot 2 + 9 = 12 - 24 + 9 = -3$ ft/s

c) The time(s) when the velocity is 0 ft/s.

$\Rightarrow$ Find t such that $v(t) = 0$

$3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0 \Rightarrow (t-1)(t-3) = 0$

$t = 1$ or 3

$f(1) = 1 - 6 + 10 = 5$

Example: Let $f(x) = x^4 - 6x^2 + 10$. Find

$f'(x) = 4x^3 - 12x + 0$

a) The equation of the tangent line at $x = 1$.

① slope $\rightarrow$ $f'(x) = 4x^3 - 12x$

slope at $x=1$ $= f'(1) = 4 - 12 = -8$

② point

③ equation

$(1, f(1)) = (1, 5)$

$y = -8x + 8 + 5 = -8x + 13$

$y - 5 = -8(x - 1)$

(slope point form on slope $= -8$, point $= (1, 5)$)

b) Find the values of x where the tangent line is horizontal.

$\Leftrightarrow$ " where $f'(x) = 0$.

$f'(x) = 4x^3 - 12x = 0$

$4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$

$4x(x - \sqrt{3})(x + \sqrt{3}) = 0$

$\Rightarrow \boxed{x = 0, \sqrt{3}, -\sqrt{3}}$

Example: The total sales of a company (in millions of dollars) t months from now are given by $S(t) = 0.015t^4 + 0.4t^3 + 3.4t^2 + 10t - 3$. Find $S(4)$ and $S'(4)$. Then, **interpret** both results.

$S'(t) = 4\cdot 0.015t^3 + 3\cdot 0.4\cdot t^2 + 2\cdot(3.4)\cdot t + 10 + 0$

$= 0.06t^3 + 1.2t^2 + 6.8t + 10$

$S(4) =$ Use calculator $= 120.84$

$S'(4) =$ " $= 60.24$

$S(4)$: The amount of sales after 4 months will be 120.84 mil\$.

$S'(4)$: Four months now on sales will increase by 60.24 mil\$/month

Marginal Cost, Revenue, and Profit - If x is the number of units of a product produced in some time interval, then

1) **total cost** = $C(x)$ and **marginal cost** = $C'(x)$

2) **total revenue** = $R(x)$ and **marginal revenue** = $R'(x)$

3) **total profit** = $P(x) = R(x) - C(x)$ and **marginal profit** = $R'(x) - C'(x)$

*Marginal cost (or revenue or profit) is the instantaneous rate of change of cost (or revenue or profit) relative to production at a given production level.

Example: A company that makes grills has a total weekly cost function (in dollars) of $C(x) = 10{,}000 + 90x - 0.05x^2$, where x is the number of grills produced.

a) Find $C(500)$ and interpret your answer.

$C(500) = 10000 + 45000 - 0.05 \cdot 250000$ $(= 12500)$

$= 42{,}500\ \$$

When you produce 500 grills then cost is 42,500$

b) Find the marginal cost at a production level of 500 grills per week. Then, interpret your result.

$C'(x) = 0 + 90 - 0.1x = -0.1x + 90$

$C'(500) = -0.1 \cdot (500) + 90 = -50 + 90 = 40.\$/\text{grill}$

When you already produce 500 grills, then producing 1 more grill will increase the cost by amount of 40.$.

c) Approximate/estimate the cost of producing 501 grills.

500 grills → costs 42,500 $

$C(500) + C'(500) = 42{,}500\$ + 40\$$

d) Approximate/estimate the cost of the 501st grill. $\boxed{= \text{Marginal Cost}}$

40$

e) Find the **exact cost** of producing the 501st grill.

$C(501) - C(500) = \text{Calculate} = 39.95\ \$$

Derivatives of Exponential and Logarithmic Functions

1. $\frac{d}{dx}e^x = e^x$
2. $\frac{d}{dx}\ln x = \frac{1}{x}$
3. $\frac{d}{dx}b^x = b^x \ln b$
4. $\frac{d}{dx}\log_b x = \frac{1}{\ln b}\left(\frac{1}{x}\right)$

$\boxed{\log_b x = \frac{\ln x}{\ln b}}$ $\quad \boxed{b = e^{\ln b}} = b^{\ln e} = 1$

$b^x = (e^{\ln b})^x = (e^{x\ln b})$

$(b^x)' = \ln b \cdot e^{x\ln b} = \ln b \cdot b^x$

$\frac{d}{dx}\log_b x = \frac{d}{dx}\frac{\ln x}{\ln b}$

$= \frac{1}{\ln b}\frac{d}{dx}\ln x$ constant

$= \frac{1}{\ln b}\cdot\frac{1}{x}$

Example: Find $\frac{dy}{dx}$ for the following. **Do not simplify your answer!**

a) $y = \log_\pi x$ $\quad \frac{dy}{dx} = \left(\begin{matrix} b=\pi \\ \text{Rule } 4\end{matrix}\right)$ $\quad \frac{dy}{dx} = \frac{1}{\ln \pi}\cdot\frac{1}{x}$

b) $y = 7.129^x$ $\quad \frac{dy}{dx} = \left(\begin{matrix}\text{Rule } 3 \\ b=7.129\end{matrix}\right) = (7.129)^x \ln(7.129)$

c) $y = 8e^x - e^8 + \frac{1}{9}\ln x$ $\quad \frac{dy}{dx} = 8e^x + 0 + \frac{1}{9}\cdot\frac{1}{x}$

d) $y = x^4 - 4^x - 3\log_6 x + 7(5^x)$ $\quad b=6$

$\frac{dy}{dx} = 4x^3 - 4^x \ln 4 - 3\cdot\frac{1}{\ln 6}\cdot\frac{1}{x} + 7\cdot 5^x \cdot \ln 5$

e) $y = \frac{x^3 - 3x(2^x) - \frac{4}{11}}{x}$

$= x^2 - 3(2^x) - \frac{4}{11}x^{-1}$

$\frac{dy}{dx} = 2x - 3(2^x\cdot\ln 2) - \frac{4}{11}\cdot(-x^{-2})$

$= 2x - (3\ln 2)\cdot 2^x + \frac{4}{11}x^{-2}.$

Note: In some cases it might be necessary to simplify a function using logarithmic properties before taking the derivative.

Example: Find $f'(x)$ if $f(x) = 5 + 7\ln\frac{6}{x^3}$.

$\ln\frac{6}{x^3} = \ln 6 - \ln x^3$

$= \ln 6 - 3\ln x$

$= 5 + 7(\ln 6 - 3\ln x)$

$f(x) = 5 + 7\ln 6 - 21\ln x.$

$f'(x) = 0 \quad 0 \quad -21\cdot\frac{1}{x} \quad = -21\cdot\frac{1}{x}$