142_18a_3.2

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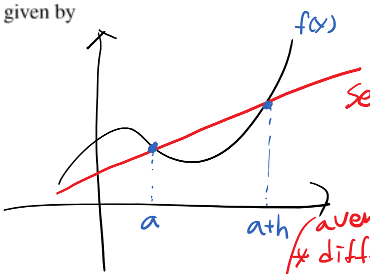
3.2 Supplement: Rates of Change

Slope of the Secant Line/Average Rate of Change

*__Slope of the Secant Line__ - A line through two points on the graph of a function is called a **secant line**. If the points $(a, f(a))$ and $(a+h, f(a+h))$ are two points on the graph of $y = f(x)$, then the slope of the secant line is given by



If x is time
$f(x)$ is distance.

$a \to a+h \Rightarrow$ Time h goes
$f(a) \to f(a+h) \Rightarrow$ distance is changed.

average velocity
* difference quotient
slope of secant line
average rate of chang

$$\frac{f(a+h) - f(a)}{(a+h) - a}$$

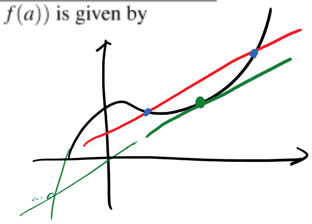
is called (average) speed. (velocity)

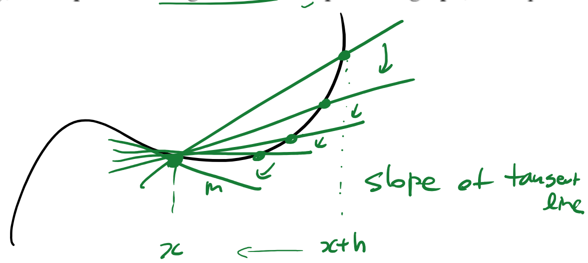
*The slope of the secant line can also be interpreted as the ______________________________ or ______________________________. Some examples of the average rate of change include...

"average" | Secant line
: finding slope

Slope of the Tangent Line/Instantaneous Rate of Change

*__Slope of the Tangent Line__ - Given $y = f(x)$, the **slope of the tangent line**, or **slope of the graph**, at the point $(a, f(a))$ is given by





Slope : $\lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x}$

NOTE: The above limit exists if and only if the slopes of the secant lines between $x = a$ and x values to the ______________ and ______________ of a approach the same value (i.e. the slope of the tangent line).

*The slope of the tangent line can also be interpreted as the ______________________________ or ______________________________. Some examples of the instantaneous rate of change include...

tangent line

x (point) $\to$ $x+h$ (virtual point) $\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x}$

Example: The revenue (in dollars) from the sale of x toasters each week is given by

$$R(x) = 50x - 2x^2$$

where $0 \leq x \leq 25$.

$R(15)$
$= 15 \cdot 50 - 2 \cdot 225$
$= 750 - 450$
$= 300$

$R(7) = 50 \cdot 7 - 2 \cdot 49$
$= 350 - 98$
$= 252.$

a) Find the change in revenue if production increases from 7 to 15 toasters each week.

$252 = R(7)$: revenue when 7 toasters are sold.
$300 = R(15)$: " 15 "
$\boxed{48} = R(15) - R(7)$: change of revenue.

b) Find the average change in revenue if production increases from 7 to 15 toasters. Then, **interpret** your answer.

$$\frac{\$}{\#} \quad \frac{R(15)-R(7)}{15-7} = \frac{48}{8} = 6 \,\$/1 \text{ toasters}$$

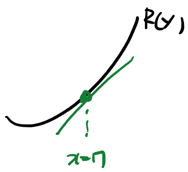
c) Find the rate of change of revenue at a production level of 7 toasters. Then, **interpret** your answer.

$$\lim_{h\to 0} \frac{R(7+h)-R(7)}{(7+h)-7} = \lim_{h\to 0}\left(\frac{50\cdot(7+h) - 2(7+h)^2 - (50\cdot 7 - 2\cdot 7^2)}{h}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{50\cdot 7} + 50\cdot h - 2(\cancel{49} + 14h + h^2) - \cancel{50\cdot 7} + \cancel{2\cdot 49}}{h}$$

$$= \lim_{h\to 0} \frac{50\cdot \cancel{h} - 28\cancel{h} - 2h^{\cancel{2}}}{\cancel{h}} = \lim_{h\to 0} \underbrace{50-28-2h}_{\text{poly}}$$

$$= \boxed{22}$$



In average, if toaster 7 ⟶ 15
Revenue increased by \$6.

At # toasters = 7, If you increase 1 more
then you can earn \$ 22.

$f(2+h) =$

Example: Suppose an object moves along the y axis so that its location is $y = f(x) = x^2 + x$ at time x, where y is in meters and x is in seconds. Find

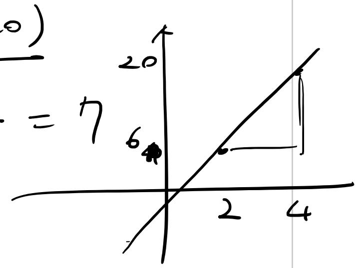
a) slope of secant line

a) The average velocity between 2 and 4 seconds.

$7\ m/s$

$(2,6) \quad (4,20)$

$$\frac{14}{2} = \frac{20-6}{4-2} = 7$$



b) The average velocity between 2 and $2+h$ seconds.

$(h+5)\ m/s$

b)

$(2,6) \quad (2+h,\ h^2+5h+6)$

$$f(2+h) = (2+h)^2 + 2+h = h^2+4h+4+2+h$$
$$= h^2+5h+6$$

slope of tangent line

c) The velocity at 2 seconds.

$$\frac{f(2+h)-f(2)}{2+h-2} = \frac{h^2+5h+\cancel{6}-\cancel{6}}{\cancel{2}+h-\cancel{2}} = \frac{h^2+5h}{h} = h+5$$

$$\lim_{h\to 0}(h+5)\ m/s = 5\ m/s$$

$$\boxed{\lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2}}$$

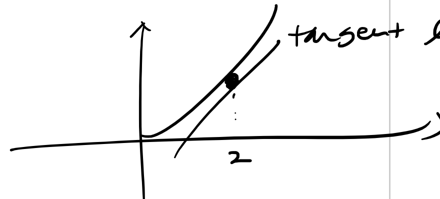
Example: Let $f(x) = 3x^2$ and find

a) The slope of the secant line between $x = 2$ and $x = 5$ (i.e. between the points $(2, f(2))$ and $(5, f(5))$).

$$m = \frac{f(5)-f(2)}{5-2} = \frac{75-12}{3} = \frac{63}{3} = 21$$

b) The equation of the tangent line at $x = 2$ (i.e. at $(2, f(2))$).

The line contains $(2, f(2))$
$= (2, 12)$



$$m = \lim_{h\to 0}\frac{f(2+h)-f(2)}{\cancel{2}+h-\cancel{2}} = \lim_{h\to 0}\frac{3(2+h)^2-3\cdot 2^2}{h}$$
$$= \lim_{h\to 0}\frac{3(\cancel{4}+4h+h^2)-\cancel{3\cdot 4}}{h}$$
$$= \lim_{h\to 0}\frac{12\cancel{h}+3h^{\cancel{2}}}{\cancel{h}} = \lim_{h\to 0}(12+3h)$$
$$= 12$$

Slope of tangent line
= limit of slope of secant line between $\boxed{x}$ and $\boxed{x+h}$

Example: The following table gives some values of a function, $f(x)$, rounded to 5 decimal places. Use the information to estimate the slope of the tangent line to $y = f(x)$ at $x = 1$.

$\boxed{\frac{1}{2}}$

x	0.98	0.99	1	1.01	1.02
$f(x)$	0.98995	0.99499	1	1.00499	1.00995

$h = 0.02$
$h = 0.01$
$h = -0.01$

$f(x+h) - f(x)$

$$\frac{f(x+h)-f(x)}{x+h-x}$$

$h=0.01$

$h=-0.01$

$h=-0.02$

① Case: $h=0.02 \Rightarrow$ find slope of secant line between $x=1$, and $x=0.98$

$$\frac{f(1)-f(0.98)}{1-0.98} = \frac{0.01005}{0.02} = \frac{1005}{2000} = 0.5xxx..$$

② Case: $h=0.01 \Rightarrow$ $\frac{f(1)-f(0.99)}{1-0.99} = \frac{0.005001}{0.01} = \frac{5001}{10000} = 0.5001.$

Example: The table below gives values of $P(t)$, the population of a small city in Texas in year t. (Midyear estimates are given.)

t	1994	1996	1998	2000	2002
$P(t)$	29,036	29,672	32,300	36,205	38,260

Find the average rate of growth from 1996 to 2000, and interpret your answer. (Round your final answer to the ~~nearest integer, if necessary.~~)

$$m = \frac{36205-29672}{2000-1996} = \frac{6533}{4}$$

$= 1633.25$

$\fallingdotseq 1633.$

(Average new residence per year. between 1996 and 2000).

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3.3 Supplement: The Derivative

The Derivative - For $y = f(x)$, we define the **derivative of f at x**, denoted by $f'(x)$, to be

$\boxed{= \text{slope of tangent line at } x}$

*If $f'(x)$ exists for each x in the open interval (a,b), then f is said to be **differentiable** over (a,b).

Interpretations of the Derivative - The derivative of a function f is a new function f'. The domain of f' is a subset of the domain of f. The derivative has various applications and interpretations, including the following:

1. **Slope of the Tangent Line** or
2. **Instantaneous Rate of Change** or
3. **Instantaneous Velocity** or

Four-step Process for Finding the Derivative $f'(x)$

Example: Use the four-step process to find $f'(x)$ if $f(x) = \sqrt{x} + 2$, and then use your result to find the equation of the tangent line of f at $x = 9$.

Step ① $f(x+h) - f(x)$

Step ② $x+h-x = h$

Step ③ : Simplify $\dfrac{f(x+h)-f(x)}{h}$

Step ④ : Take limit on $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

① $f(x+h) - f(x)$

$= \sqrt{x+h} + \cancel{2} - (\sqrt{x} + \cancel{2})$

$= \sqrt{x+h} - \sqrt{x}$

② h

③ $\dfrac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\,(\sqrt{x+h}+\sqrt{x})}$

$= \dfrac{\cancel{x}+\cancel{h}\cancel{-x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

④ $\lim_{h\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$128 = 2^7$

$16 \cdot 12 = 2^4 \cdot 4 \cdot 3$

$= 2^4 \cdot 2^2 \cdot 3$

$= 2^6 \cdot 3$

$= 2^4 \cdot 2^2 \cdot 3$
$= 2^6 \cdot 3$

Example: The height of a ball thrown upward is given by $s(t) = -16t^2 + 128t$ feet, where t is time in seconds. Use the limit definition of the derivative (i.e. the four-step process) to find the instantaneous velocity (i.e., velocity) when $t = 6$.

① $s(6+h) - s(6) = -16(6+h)^2 + 128\cdot(6+h) - (-16\cdot 6^2 + 128\cdot 6)$

$= -16(\cancel{36} + 12h + h^2) + \cancel{128\cdot 6} + 128\cdot h + \cancel{16\cdot 6^2} - \cancel{128\cdot 6}$

$= -16\cdot 12h - 16\cdot h^2 + 128\cdot h = \boxed{-64h - 16h^2}$

128
$-16\cdot 12$
$= 2^7 - 2^6\cdot 3 = 2^6(2-3) = -2^6 = -64$

② $6+h-6 = h.$ ③ $\frac{-64\cancel{h} - 16h^{\cancel{2}}}{\cancel{h}} = -64 - 16h.$

④ $\lim_{h\to 0}(-64 - 16h) = -64$. feet/sec

Nonexistence of the Derivative - The existence of a derivative at $x = a$ depends on the existence of a limit at $x = a$, that is, on the existence of

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

If the limit does not exist at $x = a$, we say the function f is **nondifferentiable at** $x = a$, or $f'(a)$ **does not exist**.

*Where does the above limit not exist (i.e. in what ways can a function f fail to be differentiable)?

Note: If f is differentiable at a, then f is continuous at a. But, if f is continuous at a, then f is not necessarily differentiable at a.