$\lim_{h\to0}(\sqrt{4+h}+\sqrt{4}) = \sqrt{\lim_{h\to0}4+h} + \sqrt{\lim_{h\to0}4} = \sqrt{4}+\sqrt{4}$
$= 2+2 = 4$

Limits of Difference Quotients - One of the most important limits in calculus is the limit of the difference quotient:

$= \lim_{h\to0}\frac{1}{(\sqrt{4+h}+\sqrt{4})} = \frac{1}{4}$

$$\boxed{\lim_{h\to0}\frac{f(a+h)-f(a)}{h}}$$

$= \lim_{h\to0}\frac{(4+h-4)}{h(\sqrt{4+h}+\sqrt{4})} = \lim_{h\to0}\frac{\cancel{h}}{\cancel{h}(\sim)}$

Example: Find the following limit for $f(x)=\sqrt{x+2}$, if it exists:

$$\lim_{h\to0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to0}\frac{(\sqrt{(2+h)+2}-\sqrt{(2)+2})}{h}\cdot\frac{(\sqrt{4+h}+\sqrt{4})}{(\sqrt{4+h}+\sqrt{4})}$$

$\lim_{h\to0}(\sqrt{2+h+2}-\sqrt{2+2}) = \sqrt{\lim_{h\to0}(2+h+2)} - \sqrt{\lim_{h\to0}(2+2)} = \sqrt{4}-\sqrt{4}$
poly over h
$= 0$

$\lim_{h\to0} h = 0$

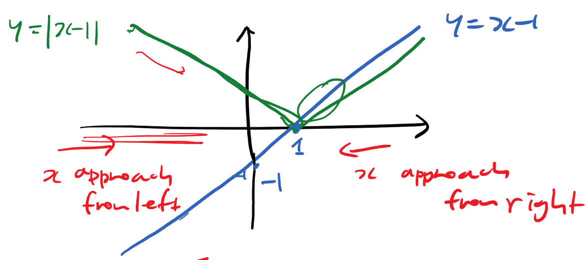
* when you see $\sqrt{} - \sqrt{}$
⇒ multiply conjugate

Example: Find the following limit for $f(x)=-x^2+3$, if it exists:

$$\lim_{h\to0}\frac{f(-5+h)-f(-5)}{h} = \lim_{h\to0}\frac{10h-h^2}{h} = \lim_{h\to0}\frac{\cancel{h}(10-h)}{\cancel{h}} \boxed{= 10}$$

$f(-5+h)-f(-5) = -(-5+h)^2+3-(-(-5)^2+3)$
$= -(\cancel{(-5)^2}-10h+h^2)\cancel{+3} + \cancel{(-5)^2}\cancel{-3}$
$= -(-10h+h^2) = 10h-h^2$

$\lim_{h\to0}(10-h) = 10-0 = 10$
poly
plug in 0



Example: Find $\lim_{x\to1}\frac{|x-1|}{x^2-1}$, if it exists.

right limit

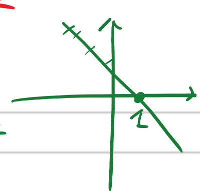
$\lim_{x\to1^+}\frac{|x-1|}{x^2-1} = \lim_{x\to1^+}\frac{x-1}{x^2-1} = \lim_{x\to1^+}\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)} = \lim_{x\to1^+}\frac{1}{x+1} = \frac{1}{2}$

left limit

$\lim_{x\to1^-}\frac{|x-1|}{x^2-1} = \lim_{x\to1^-}\frac{-\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \lim_{x\to1^-} -\frac{1}{x+1} = -\frac{1}{2}$

ex) $\lim_{x\to0}|x^2| = \lim_{x\to0}x^2$

ex) $\lim_{x\to1}|-x+1|$
from the right
$-(x+1)$

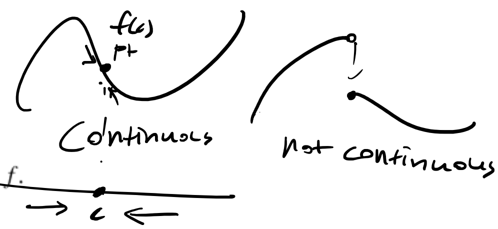


Math 142 -copyright Angela Allen, Spring 2018

from the right
$-(x+1)$
from the left
$(-x+1)$

Definition of Continuity - A function f is **continuous at a number** c if

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists $\rightarrow \leftarrow$
3. $\lim_{x \to c} f(x) = f(c)$ $\rightarrow \cdot \leftarrow$



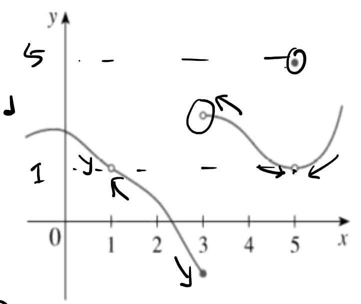
Example: The figure below shows the graph of a function f.

For what x value(s) is f discontinuous? Why?

$x=1 \rightarrow f(x)$ is undefined
$\Rightarrow$ Violate ①

$x=3 \rightarrow \lim_{x\to 3} f(x)$ is not defined
$\Rightarrow$ Violate ②

$x=5 \rightarrow f(5) \neq \lim_{x\to 5} f(x)$
$\Rightarrow$ Violate condition ③



Source: *Single Variable Calculus: Concepts & Contexts*, 3rd ed., by Stewart.

Theorem: Polynomials, rational functions, root functions, exponential functions, and logarithmic functions are continuous on their domain.

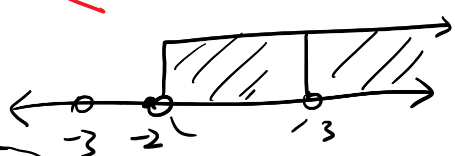
Example: Where is $f(x)$ continuous? Write your answers using interval notation.

a) $f(x) = \dfrac{\log_8(x+2) - 1}{\sqrt[5]{x^2-9}}$

$\sqrt[5]{x^2-9} \neq 0 \quad x^2-9 \neq 0 \Rightarrow (x+3)(x-3) \neq 0$
$\Rightarrow$ ① $x \neq 3, -3$

~~$\sqrt[5]{(x^2-9)}$~~ odd root $\Rightarrow$ We don't need to care.

$x+2>0$
② $x>-2$



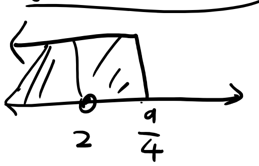
$-2<x<3$, $3<x \Rightarrow (-2,3)\cup(3,\infty)$

① $f(x)$ is combination of $\log$, rational, root function
So it is continuous at its domain. by Thm
② Find Domain.) ① check denominator $\Rightarrow x \neq -3, 3$
② " logarithm $\Rightarrow x>-2$) $\Rightarrow (-2,3)\cup(3,\infty)$

b) $f(x) = \dfrac{3^{2x-4}}{\ln(-4x+9)}$

① $\ln(-4x+9) \neq 0 \Rightarrow -4x+9 \neq 1 \Rightarrow$
② $-4x+9>0 \Rightarrow -4x>-9 \Rightarrow \boxed{x<\frac{9}{4}}$
$\quad -9 \quad -9 \qquad \overline{-4} \quad \overline{-4}$

$-4x \neq 1-9=-8$
$x \neq \frac{-8}{-4} = 2$ $\boxed{x \neq 2}$



$(-\infty,2)\cup(2,\tfrac{9}{4})$

c) $f(x) = \dfrac{e^{\frac{2x}{x-8}}}{\sqrt[3]{x^2+1}}$ → $x-8 \neq 0 \Rightarrow x \neq 8$

→ $\sqrt[3]{x^2+1} \neq 0 \Rightarrow$ No x make it 0

Domain $f(x) = (-\infty, 8) \cup (8, \infty)$

d) $f(x) = \dfrac{\sqrt[9]{x} + e^{\frac{\sqrt{1-x}}{x+5}}}{\sqrt[8]{7-4x}}$

$$\boxed{\begin{array}{l} \text{1) rational: ① } \sqrt[8]{7-4x} \neq 0 \\ \text{② } x+5 \neq 0 \\ \text{2) root: ③ } 7-4x \geq 0 \\ \text{④ } 1-x \geq 0 \end{array}}$$

domain = where f(x) continuous.

①, ③ $7-4x > 0$

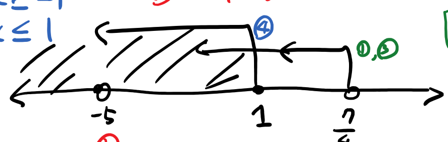
$-4x > -7$

$\boxed{x < \frac{7}{4}}$

④ $-x \geq -1$

$x \leq 1$

② $x \neq -5$



$(-\infty, -5) \cup (-5, 1]$

e) $f(x) = \dfrac{\sqrt[4]{3x-10}}{6^{\log(15-x)/(12-x)}}$ $\boxed{\text{exercise}}$

Example: Let

$$f(x)=\begin{cases}\dfrac{x-5}{x+2} & x\le -3\\ 2x^2 & -3<x\le 0\\ \dfrac{x}{x-4} & x>0\end{cases}$$

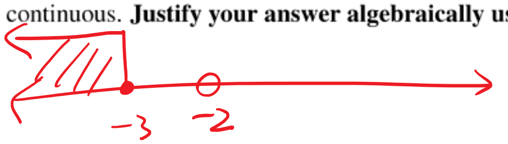
and find where f is continuous. **Justify your answer algebraically using the definition of continuity.**

intervals in piecewise function

(domain of $\frac{x-5}{x+2}$) ⇒ ① $(-\infty,-2)\cup(-2,\infty)$

domain of $2x^2$ = ② $(-\infty,\infty)$

domain of $\frac{x}{x-4}$ = ③ $(-\infty,4)\cup(4,\infty)$

① 

⇒ $(-\infty,-3]$

② $(-3,0]$

③ $(0,4)\cup(4,\infty)$

Union all of them

$(-\infty,-3]\cup(-3,0]$
$\cup(0,4)\cup(4,\infty)$
$=(-\infty,4)\cup(4,\infty)$

* Find an interval where f(x) is continuous if f(x) is piecewise

⇒ ① Find all intervals of each function's domain (without thinking (forgetting) piecewise)

② Just intersect them with intervals in piecewise function

③ Union all those remaining intervals.

Example: Let

odd ⇒ we don't need to care

$$f(x)=\begin{cases} ① \ \dfrac{\sqrt[5]{3x^2-2x+4}}{x^2-4x-12} & x<4 \\ ② \ 3x+8^{x-7} & 4<x\le 7, \\ ③ \ \dfrac{e^{\frac{3}{x-6}}}{\ln(11-x)} & x>7 \end{cases}$$

$x^2-4x-12\neq 0$

$\Rightarrow (x-6)(x+2)\neq 0$

$\boxed{x\neq 6,-2}$

$\boxed{\mathbb{R}}$

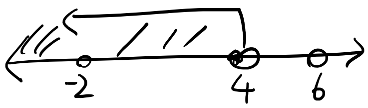
$x-6\neq 0 \Rightarrow x\neq 6$

$\ln(11-x)\neq 0 \Rightarrow 11-x\neq 1$

and find where f is continuous. **Justify your answer algebraically using the definition of continuity.**

$-x\neq -10$

$\boxed{x\neq 10}$



$11-x>0$

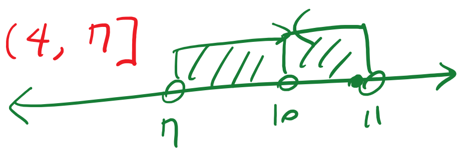
$\boxed{11>x}$

① $(-\infty,-2)\cup(-2,4)$

$11-x\neq 1$

② $(4,7]$

③



$(7,10)\cup(10,11)$

$\Rightarrow (-\infty,-2)\cup(-2,4)\cup(4,7]\cup(7,10)\cup(10,11)$

$\Rightarrow (-\infty,-2)\cup(-2,4)\cup(4,10)\cup(10,11)$