Limits: An Algebraic Approach

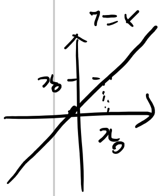
Properties of Limits - Let f and g be two functions, and assume that

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M$$

where L and M are real numbers (both limits exist). Then,

There is two function $f(x)$, $g(x)$ and, If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ exists, (L, M are real number
(* Not ∞ or $-\infty$)

You can just add, subtract multiply or division ($M \neq 0$) as usual real number.



1. $\lim_{x\to c} k = k$ for any constant k
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$
4. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$
5. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) = kL$ for any constant k
6. $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = LM$
7. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ $L > 0$ for n even

② limit of $f(x)$ when x approaches to 2 but left side

①: limit of $f(x)$ when x approaches to -2 (both side)

Note: Each of the above properties is also valid if we replace $x \to c$ by $x \to c^+$ or $x \to c^-$.

Direct Substitution Property - If ① f is a polynomial or a rational function and ② c is in the domain of f, then

$$\boxed{\lim_{x\to c} f(x) = f(c)}$$

① $\lim_{x\to -2} f(x) \neq$ ② $\lim_{x\to 2^-} f(x)$

Example: Find $\boxed{\lim_{x\to -2} \dfrac{x^2+4}{5-3x}}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

$= \lim_{x\to -2} \dfrac{f(x)}{g(x)}$

$= \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{8}{11}$

$f(x) = x^2+4$

$g(x) = 5-3x$

poly

$\lim_{x\to -2} f(x) = f(-2) = (-2)^2+4$
$= 8$

$\lim_{x\to -2} g(x) = g(-2) = 11$

Example: Find $\lim_{x\to 1} \sqrt{3x^2-1}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

$f(x) = \sqrt{3x^2-1}$

$3\cdot 1 - 1 = 2 > 0$

$\lim_{x\to 1} \sqrt{3x^2-1} = \sqrt{\lim_{x\to 1} 3x^2-1}$ $(= 3(1)^2-1)$ $= 2$ poly

By the rule

$= \sqrt{2}$

same as putting number

Example: Let $f(x) = \begin{cases} 2x & \text{if } x \geq -1 \\ x^2+3 & \text{if } x < -1 \end{cases}$ and find $\lim\limits_{x\to -1} f(x)$, if it exists.

left: $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} (x^2+3) = (-1)^2+3 = 1+3 = 4$
poly ↑ put your number

right: $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} 2x = -2$ $\neq$ the not same $\Rightarrow$ DNE
poly

Note: There are some restrictions on the limit properties. For example, property 7 (the limit of a quotient) does not apply when $\lim\limits_{x\to c} g(x) = 0$.

Limit of a Quotient - If $\lim\limits_{x\to c} f(x) = L$, $L \neq 0$, and $\lim\limits_{x\to c} g(x) = 0$, then $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ **does not exist.**

$= \frac{\lim f(x)}{\lim g(x)} \not\times \frac{L}{0}$

In other words,

D

Remember, we can numerically investigate the limit to determine the **way** in which the limit does not exist.

Indeterminate Form - If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, then $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form.**

Question: If a limit of a function is 0/0 indeterminate form, what techniques can we use to further investigate the limit?

① cancel out by factoring

② Use the conjugate $\Rightarrow$ $\frac{f(x)}{(\sqrt{x-9} - \sqrt{x+9})}$ g(x)

multiply $(\sqrt{x-9} + \sqrt{x+9})$ both num/den

we didn't do it ③ L'Hopital rule.

$\left(\frac{\#}{0}\right)$

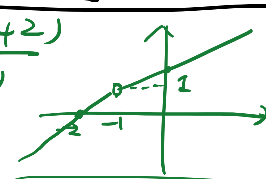
Example: Find $\lim\limits_{x\to 5} \frac{x^2-1}{x-5}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

$\lim\limits_{x\to 5} (x^2-1) = 5^2-1 = 25-1 = 24$
poly

$\lim\limits_{x\to 5} x-5 = 5-5 = 0$
poly

DNE

x $\quad x \neq -1$ $\quad \frac{x^2+3x+2}{x+1} = \frac{\cancel{(x+1)}(x+2)}{\cancel{(x+1)}}$ $\quad \frac{f(x)}{g(x)}$



$b \quad \# \quad \# \quad 0.$

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$\frac{\#}{\#}$ $\frac{\#}{0}$ $\frac{0}{0}$

Example: Find $\lim\limits_{x\to -1}\frac{x^2+3x+2}{x+1}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

$f(x) = x^2+3x+2$, $g(x) = x+1$

$\lim\limits_{x\to -1} f(x) = f(-1) = (-1)^2 - 3 + 2 = 1 - 3 + 2 = 0$ (poly)

$\lim\limits_{x\to -1} g(x) = g(-1) = -1 + 1 = 0$ (poly)

$x^2+3x+2 = (x+1)(x+2)$

$$\lim_{x\to -1}\frac{x^2+3x+2}{x+1} = \lim_{x\to -1}\frac{\cancel{(x+1)}(x+2)}{\cancel{(x+1)}} = \lim_{x\to -1}(x+2) = +1$$

Note: We were able to compute the limit of the function $f(x) = \frac{x^2+3x+2}{x+1}$ by replacing it with a simpler function $g(x) = x+2$ with the same limit. This is valid because $f(x) = g(x)$ everywhere except at $x = -1$, and when we calculate a limit we don't consider what happens when x is actually *equal* to -1.

Example: Find $\lim\limits_{x\to 4}\frac{x-4}{(x-4)^2}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

$\frac{0}{0}$

$\lim\limits_{x\to 4}(x-4) = 4-4 = 0$ (poly)

$\lim\limits_{x\to 4}(x-4)^2 = 0^2 = 0$

$$\lim_{x\to 4}\frac{(x-4)}{(x-4)^2} = \lim_{x\to 4}\frac{1}{x-4}$$

Cancel out common factor

$\lim\limits_{x\to 4} 1 = 1$

$\lim\limits_{x\to 4}(x-4) = 0$

$\frac{\#}{0}$ = Limit doesn't exists (DNE)

$\frac{(x-4)}{(x+4)(x-4)}$

Summary:

How to solve limit problem

a) Piecewise? ⇒ Check left limit and right limit. → If two limits agree ⇒ limit exists and value = one of the limit. Otherwise DNE

b) plug in x if function is rational or polynomial

- $\frac{\#}{\#}$ Done. (finished)
- $\frac{\#}{0}$ Limit does not exist.
- $\frac{0}{0}$ → algebra (Cancel out common factor ⋮)
  - $\frac{\#}{0}$ ⇒ limit doesn't exist
  - $\boxed{\#}$ Done (finished)

$\lim_{h\to 0}\left(\sqrt{4+h}+\sqrt{4}\right) = \sqrt{\lim_{h\to 0} 4+h} + \sqrt{\lim_{h\to 0} 4} = \sqrt{4}+\sqrt{4}$

$= 2+2 = 4$

__Limits of Difference Quotients__ - One of the most important limits in calculus is the limit of the difference quotient:

$$\boxed{\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

$= \lim_{h\to 0}\frac{1}{(\sqrt{4+h}+\sqrt{4})} = \frac{1}{4}$

$= \lim_{h\to 0}\frac{(4+h-4)}{h(\sqrt{4+h}+\sqrt{4})} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sim)}$

Example: Find the following limit for $f(x)=\sqrt{x+2}$, if it exists:

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\left(\sqrt{(2+h)+2}-\sqrt{(2)+2}\right)}{h}\cdot\frac{\left(\sqrt{4+h}+\sqrt{4}\right)}{\left(\sqrt{4+h}+\sqrt{4}\right)}$$

$\lim_{h\to 0}\left(\sqrt{2+h+2}-\sqrt{2+2}\right) = \sqrt{\lim_{h\to 0}(2+h+2)} - \sqrt{\lim_{h\to 0}(2+2)} = \sqrt{4}-\sqrt{4}$

$= 0$

poly over h

$\lim_{h\to 0} h = 0$

* when you see $\sqrt{} - \sqrt{}$

$\Rightarrow$ multiply conjugate

Example: Find the following limit for $f(x)=-x^2+3$, if it exists:

$$\lim_{h\to 0}\frac{f(-5+h)-f(-5)}{h}$$

Example: Find $\lim_{x\to 1}\frac{|x-1|}{x^2-1}$, if it exists.