III. Piecewise-Defined Functions

Definition: A **piecewise-defined function** is a function that is defined by different rules for different parts of its domain.

Example 4: Let

$f(-4) = (-4+1)^2 = 9$

$f(-2) = (-2+1)^2 = 1$

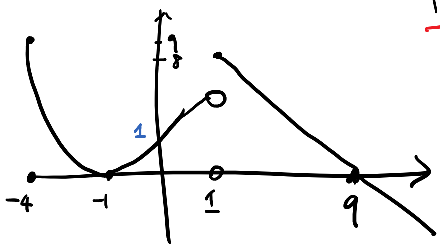
$$f(x) = \begin{cases} (x+1)^2 & -4 \le x < 1 \\ -x+9 & x \ge 1 \end{cases}$$

a) Find $f(-2)$ and $f(1)$.

b) Graph $f(x)$.

$f(1) = -1+9 = 8$

dom $[-4, \infty)$



Draw a function

① when $f(x)$ is zero: (x-intercept)

② when $x=0$ ⇒ what is $f(x)$

③ Use the usual graph of basic function (Quadratic, log, exp, linear, ...)

*Find a domain

① Denominator ⇒ check when it is zero

② $\sqrt[n]{\sim}$: n is even ⇒ $\sim$ (inside) should be non-negative

③ $\log_n(\sim)$ ⇒ $\sim$: positive

Example 5: Find the domain of the following function:

$$f(x) = \begin{cases} \dfrac{\sqrt[3]{x^2+4x-1}}{x^2-1} & 0 \le x < 2 \\ e^{12x-7}\log_2(5-x) & x > 2 \end{cases}$$

odd

① $x^2-1=0$ ⇒ $x=-1$ or 1 ⇒ $x=1$ $(-1 \notin [0,2))$

② odd root ⇒ inside can be anything.

⇒ First one: $x \in [0,1) \cup (1,2)$

$\to [0,2)$

① $\log_2(5-x)=0$ when $5-x=1$ ⇒ $x=4$

② $5-x > 0$ inside of $\log_2(\;)$ ⇒ $5 > x$

⇒ Second one: $(2,4)\cup(4,5)$

domain

$[0,1)\cup(1,2)\cup(2,4)\cup(4,5)$

$[a,b]$: $a \le x \le b$

(a,b) : $a < x < b$

Example 6: Write $f(x) = |x-8|$ as a piecewise-defined function.

$|-2| = 2$

$|3| = 3$

$$|x-8| = \begin{cases} x-8 & : \text{if } x-8 \text{ is positive} \\ -x+8 & : \text{if } x-8 \text{ is negative} \end{cases}$$

$x-8>0$ ⇒ $x>8$

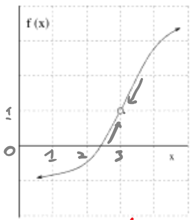
$$f(x) = \begin{cases} x-8 & : \text{if } x > 8 \\ -x+8 & : \text{if } x \le 8 \end{cases}$$

Brief Pre-Calculus Review Highly Suggested Homework Problems: It is highly recommended that you attempt to work through the Pre-Calculus Review Problems that can be found under "Pre-Calculus Review Notes and Resources" on our course page in eCampus.

3.1 Supplement: Limits

Limits: A Graphical Approach

Consider the graph of the function $f(x)$:



$f(3)$ Does not exist!
when x approach to 3
$\Rightarrow$ $f(x)$ approach to 1.
limit of $f(x)$ at $x=3$ is 1.

$\boxed{\lim_{x\to 3} f(x) = 3.}$

left right

What is $f(3)$? Not defined

What is the value of $f(x)$ as x approaches 3 from the left, i.e. $\lim_{x\to 3^-} f(x)$? 1

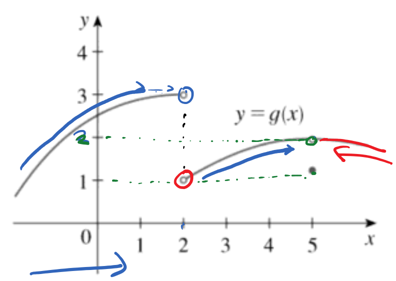
What is the value of $f(x)$ as x approaches 3 from the right, i.e. $\lim_{x\to 3^+} f(x)$? 1

Since the $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$, $\Rightarrow$ $\lim_{x\to 3} f(x)$ exists and the same as one of the one sided limit.

For a (two-sided) limit to exist, the limit from the left and the limit from the right must exist and be equal to a real number L. That is,

$$\boxed{\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L}$$

Example: The graph of a function g is shown below. Use it to state the values (if they exist) of the following:



Source: *Single Variable Calculus: Concepts & Contexts*, 3rd ed., by Stewart.

$\lim_{x\to 2} g(x)$ Does not exists

$\lim_{x\to 2^-} g(x) = 3$

$\lim_{x\to 2^+} g(x) = 1$

a) $\lim_{x\to 2} g(x)$

What is $\lim_{x\to 5} g(x) = 2$

$\lim_{x\to 5^-} g(x) = 2 = \lim_{x\to 5^+} g(x)$

b) $g(5)$ =

c) $\lim_{x\to 5} g(x)$

Note: The existence of a limit at $x = c$ has nothing to do with the function value at c. In fact, the function may or may not exist at $x = c$.

ex) $g(5) \neq \lim_{x\to 5} g(x)$

Limits: A Numerical Approach

Example: Find $\lim_{x \to 4}(3x-1)$ numerically, if it exists.

$$\begin{array}{r} 3.9 \\ \times\ 3 \\ \hline 27 \\ 9\ \ \\ \hline 11.7 \end{array}$$

$11.7 - 1 = 10.7$

$\lim_{x \to 4^+} f(x) = 11$

$\lim_{x \to 4^-} f(x) = 11$

Left

x	3.9	3.99	3.999	3.9999	...	4
$f(x)$	10.7	10.97	10.997	10.9997	$\to$	11

Right

x	4.1	4.01	4.001	...	$\to$ 4
$f(x)$	11.3	11.03	11.003	...	$\to$ 11

Example: Find $\lim_{x \to 0} \frac{1}{x^2}$ numerically, if it exists.

*Hence, we have a **vertical asymptote** at $x = 0$. We can also describe the *way* in which the limit does not exist by writing

$$\lim_{x \to 0^-} f(x) \to \infty \qquad \text{and} \qquad \lim_{x \to 0^+} f(x) \to \infty$$

Since the function is approaching ∞ from both “sides” of $x = 0$, we could also write

$$\lim_{x \to 0} f(x) \to \infty$$

If the function were approaching ∞ from one side and $-\infty$ from the other, we could not “combine” the limits to describe the behavior (we would have to write them separately).

*But, in any case, $\lim_{x \to 0} f(x)$ DOES NOT EXIST.