Section 6.5: The Fundamental Theorem of Calculus

Recall from Section 6.4:

- If f is a continuous function on $[a,b]$, then the definite integral of f from a to b can be defined as a limit of a Riemann sum for the function f:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- $\int_a^b f(x)\,dx$ gives an exact value and "counts" area **above** the x-axis **positively** and area **below** the x-axis **negatively** .
- We can attempt to use the graph of $f(x)$ to interpret $\int_a^b f(x)\,dx$ in terms of areas (i.e., use geometric shapes between $f(x)$ and the x-axis to find an exact answer when possible).
- If it is not possible to use geometric shapes to find an exact value, we can **estimate** $\int_a^b f(x)\,dx$ by using a Riemann sum.

One of the things we will learn in this section is how to evaluate a definite integral exactly (without looking at the graph of the function/using geometric shapes).

Properties of the Definite Integral- The following are properties for continuous functions f and g.

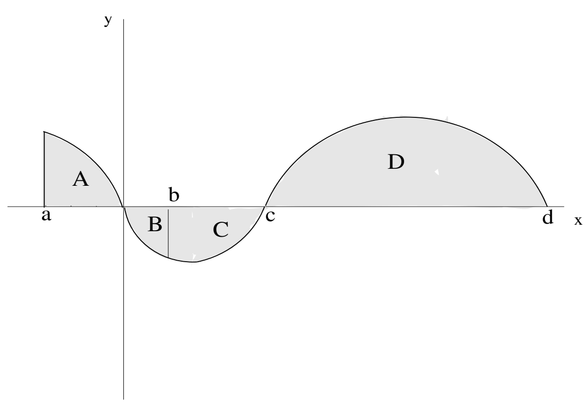
1. $\int_a^b m\,dx = m(b-a)$, where m is a constant
2. $\int_a^a f(x)\,dx = 0$
3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
4. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant
5. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

Example 1: If it is known that $\int_1^4 f(x)\,dx = 7.5$, $\int_1^4 g(x)\,dx = 21$, and $\int_4^5 g(x)\,dx = 61/3$, find

a) $\int_1^4 (4g(x) - 9f(x))\,dx$

b) $\int_1^5 (-4g(x))\,dx$

Example 2: Use the graph of $f(x)$ with the indicated areas below to answer the following.



area of A: 2.0
area of B: 1.5
area of C: 2.5
area of D: 9.5

a) Find $\int_a^c f(x)\,dx - \int_0^c 2f(x)\,dx$.

b) Find $\int_c^c 4f(x)\,dx + \int_d^b f(x)\,dx$.

We can evaluate a definite integral exactly using the Fundamental Theorem of Calculus, Part 2:

The Fundamental Theorem of Calculus, Part 2 - Suppose f is continuous on $[a,b]$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

Example 3: Evaluate the following integrals:

a) $\int_1^6 (e^x + x)\,dx$.

b) $\int_0^2 \sqrt{4x+1}\,dx$.

Using $fnInt$ on the Calculator - We can estimate the definite integral $\int_a^b f(x)\,dx$ using the calculator:

1. Type $f(x)$ into Y_1 by pressing the $y =$ button.

2. From your homescreen, press MATH and then choose option 9:fnInt(

Example 4: Find $\int_1^6 (e^x + x)\,dx$ again using $fnInt$ (from Example 3). Round your answer to 4 decimal places.

Example 5: Evaluate $\int_2^K (t^2+4)\,dt$.

Interpreting the Definite Integral as Change - We can also interpret the definite integral of a rate of change function as the **change** in its antiderivative from $x = a$ to $x = b$:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

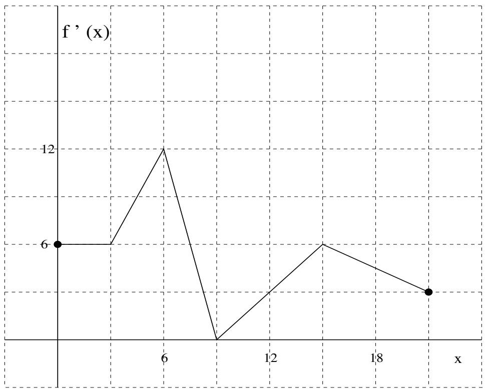
Example 6: A honeybee population starts with 200 honeybees and increases at a rate of $n'(t) = 100e^{2t}$ bees per week, where t is in weeks and $t \geq 0$.

a) Find the change in the honeybee population over the first 4 weeks. Round to the nearest integer, if necessary.

b) Find the change in the honeybee population over the second 4 week period. Round to the nearest integer, if necessary.

c) What will be the total change in the honeybee population during the seventh and eighth weeks? Round to the nearest integer, if necessary.

Example 7: Consider the graph of $f\,'(x)$ shown below. If $f(0) = 50$, find $f(9)$.



Average Value of a Continuous Function f over $[a,b]$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 8: Find the average value of $f(x) = \sqrt{x+2}$ on $[2,7]$.

Example 9: A company's marginal cost function is given by $m(x) = 0.3x^2 + 2x$ dollars per item, where x is the number of items produced. Find

a) the change in the total cost when the number of items produced increases from 10 to 20.

b) the average marginal cost over the interval $[10,20]$.