Section 6.4: The Definite Integral

Recall from Section 6.3:

- For a continuous function $f(x)$, where $f(x) \geq 0$, we can estimate the area of a region that lies under $f(x)$ from $x = a$ to $x = b$ by dividing the region into subintervals (rectangles) and adding the areas of the rectangles.
- In general, we can use any x-coordinate, x_i^*, to find the the height of the rectangle in the i^{th} subinterval.

  Using summation notation, we can write the sum of the areas of the rectangles as

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann sum.
- We can estimate the distance an object travels by estimating the area under its velocity curve using a Riemann sum (assuming the velocity function is greater than or equal to zero).

The Definite Integral

If we let the number of subintervals (n) go to infinity, then we get the actual or exact area of the region under $f(x)$ between $x = a$ and $x = b$, assuming $f(x) \geq 0$. In other words:

The above limit occurs so much, that it is given a special name and notation. We refer to this common limit as the **definite integral** of $f(x)$ from a to b and write it as

$$\int_a^b f(x)\, dx$$

Definition of a Definite Integral: Given a function $f(x)$ that is continuous on the interval $[a,b]$, we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then, the **definite integral of $f(x)$ from a to b** is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

NOTE: $\int_a^b f(x)\, dx$ "counts" area above the x-axis as positive and area below the x-axis as negative. Thus, if $f(x) \geq 0$, the definite integral represents the actual area, and if $f(x) < 0$, we say it represents the *signed* area. If the function is both positive and negative, then we say the definite integral represents the *accumulated or net* area.

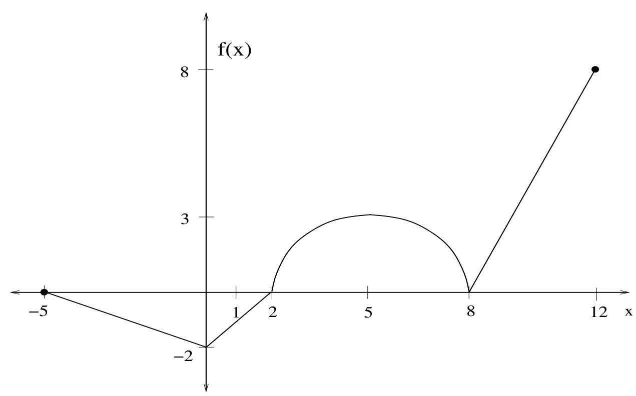
Important Notes:

1. In the notation $\int_a^b f(x)dx$, the symbol $\int$ is called an **integral sign**. It is an elongated S (since it is a limit of sums). $f(x)$ is called the **integrand** and a and b are the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The symbol dx has no official meaning by itself; $\int_a^b f(x)dx$ is all one symbol. The procedure of calculating an integral is called **integration.**

2. The definite integral $\int_a^b f(x)dx$ is a number; it does not depend on x. Recall that an indefinite integral, $\int f(x)\,dx$, represents a family functions.

Example 1: Use the graph of $f(x)$ below to find the following. Note that the graph consists of three straight lines and a semicircle.

a) $\displaystyle\int_{-5}^{2} f(x)\,dx$



b) $\displaystyle\int_{0}^{8} f(x)\,dx$

c) $\displaystyle\int_{5}^{12} f(x)\,dx$

Example 2: Evaluate each of the following by interpreting the definite integral in terms of areas.

a) $\int_2^5 8\,dx$

b) $\int_0^4 (x-1)\,dx$

Question: What do we do if we cannot use geometric shapes between $f(x)$ and the x-axis to find $\int_a^b f(x)\,dx$ exactly?

Example 3: Use a midpoint sum with $n = 3$ to estimate $\int_{-1}^{2} (x^2 - 1)\,dx$.

Note: In Section 6.5, we will learn how to evaluate a definite integral exactly without using a graph/geometric shapes!