5.3 Supplement: Limits at Infinity (and Infinite Limits)

Infinite Limits

Recall from section 3.1 we concluded that a limit such as $\lim_{x \to 4} \frac{1}{x-4}$ does not exist since

*However, we can indicate this kind of behavior (the **way** in which this limit does not exist) by using the notation

*Thus, the notation $\lim_{x \to a} f(x) \to \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a (on either side of a) but not equal to a.

Vertical Asymptotes - The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following is true:

- $\lim_{x \to a} f(x) \to \infty$
- $\lim_{x \to a} f(x) \to -\infty$
- $\lim_{x \to a^-} f(x) \to \infty$
- $\lim_{x \to a^+} f(x) \to \infty$
- $\lim_{x \to a^-} f(x) \to -\infty$
- $\lim_{x \to a^+} f(x) \to -\infty$

Question: How do we find and describe the behavior near vertical asymptotes? How do we find holes?

Example: Find $\lim_{x\to 1} \dfrac{x^2+4x+3}{x^2-1}$ **algebraically**, if it exists. If the limit does not exist, use limits to describe the **way** in which it does not exist.

Example: Find any holes and/or vertical asymptotes of the function $f(x) = \dfrac{x^2+5x+6}{x^2-x-12}$ **algebraically**. If there are vertical asymptotes, use limits to describe the behavior near each asymptote.

Limits at Infinity of Polynomials

Example: Describe the end behavior of $p(x) = 3x^3 - 500x^2$. In other words, find $\lim_{x\to -\infty} p(x)$ and $\lim_{x\to\infty} p(x)$.

Limits at Infinity of Rational Functions

Consider the graph of $f(x) = \dfrac{1}{x-4}$ again below. As x gets larger (or smaller), we see that the values of $f(x)$ get closer to 0.

*Symbolically, we write

Horizontal Asymptotes - The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

Question: How do we find horizontal asymptotes?

Calculating Limits at Infinity of Rational Functions

*If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Thus, to calculate limits at infinity (i.e. find horizontal asymptotes), we will...

Example: Find the horizontal asymptotes, **algebraically**, of the curve $y = \dfrac{2x^2+x-1}{x^4+x-2}$, if they exist.

Example: Find the horizontal asymptotes, **algebraically**, of the curve $y = \dfrac{2x^5 + x - 1}{x^2 + x - 2}$, if they exist.

Example: Find the horizontal asymptotes, **algebraically**, of the curve $y = \dfrac{2x^2 + x - 1}{3x^2 + x - 2}$, if they exist.