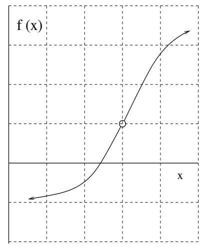
3.1 Supplement: Limits

Limits: A Graphical Approach

Consider the graph of the function $f(x)$:



What is $f(3)$?

What is the value of $f(x)$ as x approaches 3 from the left, i.e. $\lim_{x\to 3^-} f(x)$?

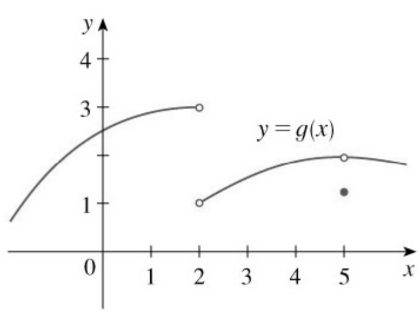
What is the value of $f(x)$ as x approaches 3 from the right, i.e. $\lim_{x\to 3^+} f(x)$?

Since the $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$,

For a (two-sided) limit to exist, the limit from the left and the limit from the right must exist and be equal to a real number L. That is,

$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Example: The graph of a function g is shown below. Use it to state the values (if they exist) of the following:



Source: *Single Variable Calculus: Concepts & Contexts*, 3rd ed., by Stewart.

a) $\lim_{x\to 2} g(x)$

b) $g(5)$

c) $\lim_{x\to 5} g(x)$

Note: The existence of a limit at $x = c$ has nothing to do with the function value at c. In fact, the function may or may not exist at $x = c$.

Limits: A Numerical Approach

Example: Find $\lim_{x \to 4}(3x-1)$ numerically, if it exists.

Example: Find $\lim_{x \to 0} \frac{1}{x^2}$ numerically, if it exists.

*Hence, we have a **vertical asymptote** at $x = 0$. We can also describe the *way* in which the limit does not exist by writing

$$\lim_{x \to 0^-} f(x) \to \infty \qquad \text{and} \qquad \lim_{x \to 0^+} f(x) \to \infty$$

Since the function is approaching ∞ from both "sides" of $x = 0$, we could also write

$$\lim_{x \to 0} f(x) \to \infty$$

If the function were approaching ∞ from one side and $-\infty$ from the other, we could not "combine" the limits to describe the behavior (we would have to write them separately).

*But, in any case, $\lim_{x \to 0} f(x)$ DOES NOT EXIST.

Limits: An Algebraic Approach

Properties of Limits - Let f and g be two functions, and assume that

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M$$

where L and M are real numbers (both limits exist). Then,

1. $\lim_{x\to c} k = k$ for any constant k
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L+M$
4. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L-M$
5. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x) = kL$ for any constant k
6. $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = LM$
7. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$ if $M \neq 0$
8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ $L > 0$ for n even

Note: Each of the above properties is also valid if we replace $x \to c$ by $x \to c^+$ or $x \to c^-$.

Direct Substitution Property - If f is a polynomial or a rational function and c is in the domain of f, then

$$\lim_{x\to c} f(x) = f(c)$$

Example: Find $\lim_{x\to -2} \frac{x^2+4}{5-3x}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

Example: Find $\lim_{x\to 1} \sqrt{3x^2-1}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

Example: Let $f(x) = \begin{cases} 2x & \text{if } x \geq -1 \\ x^2+3 & \text{if } x < -1 \end{cases}$, and find $\lim_{x \to -1} f(x)$, if it exists.

Note: There are some restrictions on the limit properties. For example, property 7 (the limit of a quotient) does not apply when $\lim_{x \to c} g(x) = 0$.

Limit of a Quotient - If $\lim_{x \to c} f(x) = L$, $L \neq 0$, and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ **does not exist**.

In other words,

Remember, we can numerically investigate the limit to determine the **way** in which the limit does not exist.

Indeterminate Form - If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form.**

Question: If a limit of a function is 0/0 indeterminate form, what techniques can we use to further investigate the limit?

Example: Find $\lim_{x \to 5} \frac{x^2-1}{x-5}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

Example: Find $\lim_{x \to -1} \frac{x^2+3x+2}{x+1}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

Note: We were able to compute the limit of the function $f(x) = \frac{x^2+3x+2}{x+1}$ by replacing it with a simpler function $g(x) = x+2$ with the same limit. This is valid because $f(x) = g(x)$ everywhere except at $x = -1$, and when we calculate a limit we don't consider what happens when x is actually *equal* to -1.

Example: Find $\lim_{x \to 4} \frac{x-4}{(x-4)^2}$, if it exists. If it does not exist, also use limits to describe the *way* in which it does not exist.

Summary:

<u>**Limits of Difference Quotients**</u> - One of the most important limits in calculus is the limit of the difference quotient:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Example: Find the following limit for $f(x)=\sqrt{x+2}$, if it exists:

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

Example: Find the following limit for $f(x)=-x^2+3$, if it exists:

$$\lim_{h\to 0}\frac{f(-5+h)-f(-5)}{h}$$

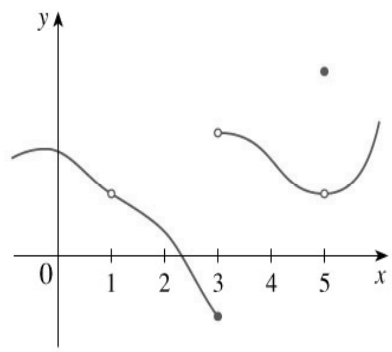
Example: Find $\lim_{x\to 1}\frac{|x-1|}{x^2-1}$, if it exists.

Definition of Continuity - A function f is **continuous at a number** c if

1. $f(c)$ is defined
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

Example: The figure below shows the graph of a function f.

For what x value(s) is f discontinuous? Why?



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Theorem: Polynomials, rational functions, root functions, exponential functions, and logarithmic functions are continuous on their ________________.

Example: Where is $f(x)$ continuous? Write your answers using interval notation.

a) $f(x) = \dfrac{\log_8(x+2)-1}{\sqrt[5]{x^2-9}}$

b) $f(x) = \dfrac{3^{2x-4}}{\ln(-4x+9)}$

c) $f(x) = \dfrac{e^{\frac{2x}{x-8}}}{\sqrt[3]{x^2+1}}$

d) $f(x) = \dfrac{\sqrt[9]{x} + e^{\frac{\sqrt{1-x}}{x+5}}}{\sqrt[8]{7-4x}}$

e) $f(x) = \dfrac{\sqrt[4]{3x-10}}{6^{\log(15-x)/(12-x)}}$

Example: Let

$$f(x) = \begin{cases} \dfrac{x-5}{x+2} & x \leq -3 \\ 2x^2 & -3 < x \leq 0 \\ \dfrac{x}{x-4} & x > 0 \end{cases},$$

and find where f is continuous. **Justify your answer algebraically using the definition of continuity.**

Example: Let

$$f(x) = \begin{cases} \dfrac{\sqrt[5]{3x^2 - 2x + 4}}{x^2 - 4x - 12} & x < 4 \\ 3x + 8^{x-7} & 4 < x \leq 7 \\ \dfrac{e^{\frac{3}{x-6}}}{\ln(11 - x)} & x > 7 \end{cases},$$

and find where f is continuous. **Justify your answer algebraically using the definition of continuity.**