RESEARCH STATEMENT: LOCAL COHOMOLOGY, MULTIGRADINGS AND POLYHEDRAL COMBINATORICS

BYEONGSU YU

1. Introduction

My area of study is combinatorial commutative algebra. The goal of combinatorial commutative algebra is to study the interplay between commutative algebra and various subfields of combinatorics such as enumerative combinatorics and discrete geometry. Among the central objects in combinatorial commutative algebra are square-free monomial ideals and semigroup rings. In this context, the Cohen–Macaulay property, an important notion from commutative algebra, has strong combinatorial relevance [Hoc72, Sta75].

Monomial ideals in affine semigroup rings, which generalize both monomial ideals in polynomial rings and affine semigroup rings, are central objects in my research. The *affine semigroup* $Q = \mathbb{N}\mathcal{A}$ is the set of non-negative integer combinations of elements of $\mathcal{A} := \{\alpha_1, \cdots, \alpha_n\} \subset \mathbb{Z}^d$. The *affine semigroup ring* over a field $\mathbb{K}$ associated with $\mathcal{A}$ is defined as $\mathbb{K}[Q] := \mathbb{K}[\mathbf{t}^{\alpha_1}, \cdots, \mathbf{t}^{\alpha^n}] \subset \mathbb{K}[t_1^{\pm 1}, \cdots, t_d^{\pm 1}]$ with notation $\mathbf{t}^{\alpha_i} := t_1^{\alpha_{i,1}} \cdots t_d^{\alpha_{i,d}}$. Affine semigroup rings have been actively studied by algebraic geometers and combinatorialists mainly due to their connection with toric varieties [Ful93, MS05].

A basic question inspired by [Hoc77] is to characterize which of the quotients of affine semigroup rings by monomial ideals are Cohen–Macaulay. Multi-gradings and polyhedral techniques are two of my main tools in this study. Affine semigroup rings are $\mathbb{Z}^d$-graded by construction. Furthermore, the polyhedral cone $\mathbb{R}_{\geq 0}Q := \{\sum_{i=1}^d c_i\alpha_i : c_i \in \mathbb{R}_{\geq 0}\}$ and its face structure are important when studying monomial ideals in $\mathbb{K}[Q]$ [MS05]. On the one hand, polynomial rings are a special case of normal affine semigroup rings, for which the underlying polyhedral cones are simplices. On the other hand, monomial ideals in affine semigroup rings can be identified as sums of toric ideals and monomial ideals in polynomial rings. Therefore, it seems natural to generalize known results for monomial ideals in polynomial rings to affine semigroup rings.

One of my contributions is generalizing standard pairs from [STV95] to the semigroup context [MY20], as well as extending the Ishida complex [Ish88] to calculate local cohomology of $\mathbb{K}[Q]$-module with support on general radical monomial ideals [MY21]. Using these two tools, I have given a combinatorial charaterization for the Cohen–Macaulay quotients of affine semigroup rings by monomial ideals [MY21]. This criterion involves vanishing homology of finitely many polyhedral cell complexes. Further work in progress investigates local cohomology for lattice ideals as well as combinatorially characterizing the Gorenstein property of quotients of affine semigroup rings by monomial ideals.

2. Current result

2.1. **Monomial ideals and polyhedral geometry.** Monomial ideals in affine semigroup rings are closely related to polyhedral geometry. For example, there is an order-reversing isomorphism between the poset $\operatorname{Spec}_{\text{mon}} \mathbb{K}[Q]$ of all prime monomial ideals of $\mathbb{K}[Q]$ and the face lattice $\mathcal{F}(\mathbb{R}_{\geq 0}Q)$ of the cone $\mathbb{R}_{\geq 0}Q$ [MS05]. The set $\mathcal{F}(Q) := \{\{\alpha : x^\alpha \notin \mathfrak{p}\} \subset Q : \mathfrak{p} \in \operatorname{Spec}_{\text{mon}} \mathbb{K}[Q]\}$ is called the *face lattice* of Q since the isomorphism $\operatorname{Spec}_{\text{mon}} \mathbb{K}[Q] \cong \mathcal{F}(\mathbb{R}_{\geq 0}Q)$ is induced by the span of the degrees of standard monomials of a monomial prime ideal over $\mathbb{R}_{\geq 0}$. This set coincides with $\{F \cap Q : F \in \mathcal{F}(\mathbb{R}_{\geq 0}Q)\}$ [MS05]. Likewise, all radical monomial ideals correspond to polyhedral subcomplexes of $\mathbb{R}_{\geq 0}Q$. These facts justify the notation I_Δ for a radical monomial ideal, where Δ is a polyhedral subcomplex of $\mathbb{R}_{\geq 0}Q$.

On the other hand, the map $\mathbb{R}_{\geq 0}(-)$ sending an affine semigroup Q to the cone $\mathbb{R}_{\geq 0}Q$ is not injective but surjective. In other words, there are affine semigroups that do not contain all integral points of $\mathbb{R}_{\geq 0}Q$. Thankfully, the *holes*, integral points of $\mathbb{R}_{\geq 0}Q$ not belonging to Q, can be decomposed into a union of translated faces [HTY09, Kat15, MY21]. All of these connections naturally lead me to study monomial ideals as polyhedral-geometric objects.

2.2. **Standard pairs as polyhedral geometric objects.** The notion of *standard pair* was introduced in [STV95] as a combinatorial structure to calculate bounds on geometric multiplicities. This can be generalized for monomial ideals of affine semigroup rings [MY20]. For a monomial ideal I of $\mathbb{K}[Q]$, a *proper pair* $(\mathbf{t}^\alpha, F)$ for some $\mathbf{t}^\alpha \in \mathbb{K}[Q]$ and $F \in \mathcal{F}(Q)$ is a pair satisfying $\mathbf{t}^{\alpha+\varphi} \notin I$ for any $\varphi \in F$. We give an order between proper pairs $(\mathbf{t}^\alpha, F) > (\mathbf{t}^\beta, G)$ if $\{\mathbf{t}^{\alpha+\varphi} : \varphi \in F\} \supset \{\mathbf{t}^{\beta+\varphi} : \varphi \in G\}$. The set of *standard pairs* $\operatorname{Std}(I)$ of I is the set of maximal proper pairs of I. Moreover, we say that a pair $(\mathbf{t}^\alpha, F)$ divides $(\mathbf{t}^\beta, F)$ if there exists $\gamma \in Q$ such that $(\mathbf{t}^{\alpha+\gamma}, F) < (\mathbf{t}^\beta, F)$. The set of *overlap classes* $\overline{\operatorname{Std}}(I)$ of I is defined as $\operatorname{Std}(I)/\sim$ where two pairs of $(\mathbf{t}^\alpha, F), (\mathbf{t}^\beta, F)$ are equivalent if they divide each other.

Standard pairs and overlap classes are useful when studying a monomial ideal I of an affine semigroup ring. For example, I is primary if and only if all standard pairs of I share the same face. Moreover, I is irreducible if and only if it is primary and has a unique overlap class that is maximal with respect to divisibility. Thus, the number of maximal overlap classes of I is the same as the number of components of its irredundant irreducible decomposition [MY20]. Moreover, there are finitely many standard pairs and overlap classes.

2.3. **Generalized Ishida complex.** Matusevich and I have extended the *Ishida complex* to calculate the local cohomology of $\mathbb{K}[Q]$-module supported on radical monomial ideals. Originally, the Ishida complex calculates such cohomology supported on the maximal graded ideal only [Ish88, BH93, ILL$^+$07]. Indeed, we observed that the chain complex used in the original Ishida complex is not $\mathcal{F}(\mathbb{R}_{\geq 0}Q)$ but the unbounded rays of the convex hull of the maximal monomial ideal. More precisely, let J be a monomial ideal of $\mathbb{K}[Q]$. Let T be a transverse section of the convex hull $\operatorname{conv}(\sqrt{J})$ of $\sqrt{J}$ as a subset of $\mathbb{R}^d$ containing all unbounded faces [Zie95]. The *generalized Ishida complex* over J is the chain complex

$$L^\bullet : 0 \to L^0 = \mathbb{K}[Q] \xrightarrow{\delta} L^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} L^d \to 0$$

where $L^i = \oplus_{F \in \mathcal{F}(T)^i} \mathbb{K}[Q]_{\hat{F}}$ is a direct sum of all localizations of $\mathbb{K}[Q]$ by a monomial prime ideal whose corresponding face $\hat{F}$ is the minimal face of Q containing $F \cap Q$ over all i-dimensional faces of T. Also, $\delta : L^i \to L^{i+1}$ is a direct sum of all canonical maps of localizations between components of L^i and L^{i+1} with an appropriate choice of signs, determined by the orientation of T. In this case, the local cohomology $H^i_J(M)$ of M is isomorphic to $H^i(L^\bullet \otimes_{\mathbb{K}[Q]} M)$ for any $\mathbb{K}[Q]$-module M [MY21].

2.4. **Hochster-type formula for the Hilbert series of the local cohomology.** One of our goals was to produce a Hochster-type formula for the Hilbert series of local cohomology in this context. Fortunately, the *standard pair topology* allows us to classify the graded parts of the generalized Ishida complex that share the same polyhedral chain complex. Indeed, overlap classes of a monomial ideal I with face F are restrictions of overlap classes of the localization $(\mathbb{K}[Q]/I)_F$ by a monomial prime ideal $\mathfrak{p}_F := \{\mathbf{t}^\alpha : \alpha \in Q \setminus F\}$ to $\mathbb{K}[Q]/I$. In other words, the set of all overlap classes over all localizations cover the *standard monomial space* $\operatorname{stdm}(I) = \{\mathbf{t}^\alpha : \mathbf{t}^\alpha \in (\mathbb{K}[Q]/I)_F$ for some $F \in \mathcal{F}(Q)\}$ consisting of all monomials appearing as standard monomials for some localization. Hence, $\operatorname{stdm}(I)$ enriches a topology generated by overlap classes as open sets. Since the number of standard pairs (and equivalence classes) is always finite, the topology is finite. We saw that two monomials whose degrees are in the same minimal open set appear on the same localizations, thus that they have the same graded parts of Ishida complex. We call a minimal open set in this topology a *grain* and a set of faces of T whose corresponding localizations contain a given grain the *chaff* of the grain. Since grains partition $\operatorname{stdm}(I)$, chaffs determine all the graded parts of the Ishida complex. Hence,

$$\operatorname{Hilb}(H^i_J(\mathbb{K}[Q]/I, \mathbf{t})) = \sum_{\sigma \in \operatorname{chaff}(I)} \dim_{\mathbb{K}} H^i(\sigma, \mathbb{K}) \left(\sum_{\alpha \in G_\sigma \in \operatorname{grain}(I)} \mathbf{t}^\alpha \right)$$

where $\operatorname{chaff}(I)$ and $\operatorname{grain}(I)$ are sets of all chaffs or grains of I respectively. Since grains are intersections of translations of faces of Q, $\sum_{\alpha \in G_\sigma \in \operatorname{grain}(I)} \mathbf{t}^\alpha$ can be replaced with a rational function.

2.5. **Cohen–Macaulayness criterion for the quotients by monomial ideals.** The Hochster-type formula from the standard pair topology gives a Cohen–Macaulayness criterion for a quotient of an affine semigroup by a monomial ideal. It is a celebrated result that Stanley-Reisner rings are Cohen–Macaulay if and only if their corresponding simplicial complex Δ has links with vanishing non-top homology [Rei76]. Likewise, quotients of affine semigroup rings by monomial ideals are Cohen–Macaulay if and only if their chaffs have vanishing non-top homology [MY21]. In other words, the chaffs play the same role as links in the Stanley-Reisner case. This result generalizes the combinatorial Cohen–Macaulayness criterion for affine semigroup rings, given in [TH86] and reproved in [MY21].

2.6. **Duality between local cohomologies of Stanley-Reisner ring.** Our Hochster-type formula for quotients of affine semigroup rings elucidates a hidden duality between the local cohomologies of the Stanley-Reisner rings. To see this, fix a Stanley-Reisner ring $\mathbb{K}[\mathbf{t}]/I_\Delta$ for a monomial radical ideal I_Δ of the polynomial ring $\mathbb{K}[\mathbf{t}]$ corresponding to a simplicial complex Δ. Then Reisner's criterion can be reformulated as follows; for each face F of d-simplex, there exists a unique grain G_F whose graded part of the Ishida complex with maximal ideal support is equal to the link of F in Δ if $F \in \Delta$, or 0 if $F \notin \Delta$ and chaffs from all other grains are acyclic [MY21]. Thus, the Hilbert series of $H^\bullet_{\mathfrak{m}}(\mathbb{K}[\mathbf{t}]/I_\Delta)$ can be decomposed as a finite sum over grains G_F for each $F \in \Delta$. On the other hands, grains of the Ishida complex of $\mathbb{K}[\mathbf{t}]$ supported on I_Δ are equal to sets of lattice points from orthants of $\mathbb{K}[\mathbf{t}]$ added from localizations by a face F of d-simplex. Therefore we may label each grain as $\mathfrak{r}_F$ for each F in the d-simplex. We find that for each F in the d-simplex,

$$H^\bullet_{\mathfrak{m}}(\mathbb{K}[\mathbf{t}]/I_\Delta)_\alpha \cong H^{\dim \mathbb{K}[\mathbf{t}] - \bullet}_{I_\Delta}(\mathbb{K}[\mathbf{t}])_\beta$$

for any $\alpha \in G_F$ and $\beta \in \mathfrak{r}_{F^c}$ [MY21].

Indeed, [Hun07, Ric] mentioned that $H^\bullet_{\mathfrak{m}}(\mathbb{K}[\mathbf{t}]/I_\Delta) = 0$ if and only if $H^{\dim \mathbb{K}[\mathbf{t}] - \bullet}_{I_\Delta}(\mathbb{K}[\mathbf{t}]) = 0$ for the maximal ideal $\mathfrak{m}$ and a radical monomial ideal I_Δ, which is a corollary of this duality. Moreover, this result holds for any normal affine semigroup ring whose polyhedral cone is simplicial.

2.7. **Computational package.** All results above are constructive in the sense that algorithms computing them exist. For example, Matusevich and I devised an algorithm to find standard pairs of a monomial ideal [MY20]. Using this, I implemented a `SageMath` package `StdPair` to calculate algebraic invariants of an affine semigroup and its (monomial) ideals symbolically, such as standard pairs, associated primes, multiplicity, minimal generators, and primary decomposition [Yu20].

Moreover, we observed that transverse sections of the convex hull of ideals are polytopes generated by "cutting" faces out of other polytopes. These are all combinatorially determined regardless of their geometric realization [Yu22]. Using this, we suggest an algorithm to calculate our Hochster-type formula for a quotient of an affine semigroup ring. This generalizes an algorithm calculating the local cohomology of modules over normal affine semigroup rings [HM05].

3. Future studies

3.1. **Finding a combinatorial Cohen–Macaulayness criterion for special binomial ideals.** As we saw in Section 2.5 we have a combinatorial Cohen–Macaulayness criterion for sum of toric ideals and monomial ideals. Moreover, it seems that the Ishida complex for computing local cohomology can be extended to lattice binomial ideals. Indeed, for a lattice ideal I_L corresponding to an unsaturated lattice L whose associated toric ideal is $I_{L_{\text{sat}}}$ where L_{sat} is the saturation of L, the Ishida complex from the polyhedral structure of the affine semigroup ring $\mathbb{K}[Q] \cong \mathbb{K}[\mathbf{t}]/I_{L_{\text{sat}}}$ calculates the local cohomology of $\mathbb{K}[\mathbf{t}]/I_L$. Using this we hope to calculate the Hilbert series of the local cohomology of the quotients by lattice ideals $\mathbb{K}[\mathbf{t}]/I_L$ and obtain a Hochster-type formula. Our final goal is to give a combinatorial Cohen–Macaulayness criterion for cellular binomial ideals, which generalize lattice ideals.

3.2. **Finding a combinatorial Gorenstein criterion for quotients of affine semigroup rings.** There are four important classes of Noetherian graded rings with the unique homogeneous maximal ideals, which form the following chain of inclusions: Cohen–Macaulay rings $\supset$ Gorenstein rings $\supset$ complete intersection rings $\supset$ regular rings. For each type of ring, except Gorenstein rings, there is a combinatorial characterization discerning whether a certain quotient $\mathbb{K}[Q]/I$ of affine semigroup ring by a monomial ideal has the given property or not. For example, $\mathbb{K}[Q]/I$ is regular if and only if $\mathbb{K}[Q]/I$ has only one standard pair isomorphic to $\mathbb{N}^d$ for some d [Mat89][Theorem 14.4]. Also, [FMS97] and the $\mathbb{Z}Q$-graded Koszul complex over $\mathbb{K}[Q]/I$ give such a combinatorial characterization of complete intersection rings among affine semigroup rings or quotients of them by monomial ideals.

Hence, it seems natural to ask for a combinatorial Gorenstein criterion for quotients of affine semigroup rings by monomial ideals. Currently, combinatorial characterizations of Gorenstein affine semigroup rings [TH86, BG09] and Gorenstein Stanley-Reisner rings [Sta77, Hoc77] are known. The local cohomology of canonical modules of quotients of affine semigroup rings may answer this question. The challenge of this problem is to find a finite decomposition of the canonical module compatible with the Ishida complex.

3.3. **Characterizing local cohomology modules with infinite dimensional socles.** The *support* of a graded module M is a set of multidegrees d whose corresponding graded piece M_d is nonzero. The standard pair topology shows that the supports of local cohomology modules $H^\bullet_J(\mathbb{K}[Q]/I)$ are covered by grains whose chaff is non-exact, i.e. has topological holes. The supports of $H^\bullet_J(\mathbb{K}[Q]/I)$ are unions and set differences of sets of lattice points in polyhedral objects, since grains are exactly those sets. On the other hand, the degrees of the socle of a module M can be regarded as lattice points on faces whose outer normal vectors equal the generators of Q. Hence, one may ask whether one can find a combinatorial condition to determine whether $H^\bullet_J(\mathbb{K}[Q]/I)$ has an infinite-dimensional socle [MS05][Problem 13.18]. This problem was inspired by Hartshorne's counter-example for Grothendieck's conjecture.

3.4. **Class groups of non-normal toric varieties.** Standard pairs may help to study class groups of non-normal toric varieties. Indeed, Matusevich and I constructed *void pairs* which organize all non-lattice points of the polyhedral cone $\mathbb{R}_{\geq 0}Q$ in terms of translations of faces [MY21]. Void pairs might allow one to calculate monomial fractional ideals of non-normal affine semigroup rings. If this was true, then we could calculate the class group of non-normal toric varieties by generalizing the fact that the class group of normal toric varieties is the direct sum of the class group of the field and that of the corresponding normal monoid [BG09][Theorem 4.60].

3.5. **Irreducible resolution of quotients of non-normal affine semigroup rings.** An ideal W of $\mathbb{K}[Q]$ is *irreducible* if W cannot be expressed as an intersection of two distinct ideals [MS05]. The *irreducible resolution* of a module M is an exact sequence $0 \to M \to \overline{W}^0 \to \cdots$ such that each $\overline{W}^i$ is a direct sum of quotients of $\mathbb{K}[Q]$ by irreducible ideals. Unless Q is normal, there is no known algorithm for constructing irreducible resolutions. Although an algorithm for constructing irreducible decomposition of normal affine semigroup rings was provided in [HM05], this is currently not implemented. A general algorithm was provided in [MY20] which is implemented in [Yu20], but irreducible resolutions are not yet developed in this context. Void pairs introduced in Section 3.4 seem to give a promising approach towards finding an implementable algorithm for irreducible resolutions of monomial ideals over non-normal affine semigroup rings for `Macaulay2` or `SageMath`.

3.6. **Classifying acyclic chaffs using hyperplane arrangements.** Chaffs can be described using the *poset of regions* generated by hyperplanes of $\mathbb{R}_{\geq 0}Q$. The *poset of regions* is the poset of connected components of the complements of the union of hyperplanes of $\mathbb{R}_{\geq 0}Q$ with a partial order generated by half-spaces of the hyperplanes. The face lattice of $\mathbb{R}_{\geq 0}Q$ is embedded in this poset [BEZ90]. For example, every grain of the zero ideal of an affine semigroup ring has chaff equivalent to the intersection of the upper set of the poset of regions and the face lattice of the cone of the corresponding affine semigroup.

Despite the fact that the situation when the poset of regions is a lattice is well studied [Rea03b, Rea16, DHMP20, Rea03a], it is still unclear whether the intersection between the upper set of the poset of regions and the face lattice of cone as a sub-lattice of the poset of regions is acyclic or not. In the spirit of the characterization of f-vectors of simplicial complexes [Duv94, Sta93] in relation to Kalai's conjecture [BK88], this question may shed lights on the combinatorial decomposition of polyhedral complexes.

3.7. **Software Development.** I plan to distribute libraries for calculating algebraic properties of affine semigroup rings and their quotients. Toward this goal, an algorithm the Hilbert series of local cohomologies of affine semigroup rings over a radical monomial ideal will be integrated in the package `StdPair` [MY20]. Moreover, this package will be redistributed as a C++ library so that not only `SageMath` users but also `Macaulay2` users may benefit from `StdPair`.

3.8. **Other interests.**

3.8.1. *Interdisciplinary research.* Although my main interest lies in combinatorial commutative and homological algebra and their relative fields, I also have an interest on working with statisticians or scientists in other disciplines. Recently, Kisung You and I suggested a gradient-free dimension reduction algorithm finding a linear projection preserving the persistence homology [YY21]. In this paper, we suggested measures showing how many portions of the filtration of Rips complexes over the original data are quasi-isomorphic (resp. homotopy equivalent) to the filtration of Rips complexes over the projected data. These measures can be used for evaluating the effectiveness of dimensionality reduction using linear projection with topological data analysis.

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