Motivation: To study schemes, an important and fun generalization of quasiprojective varieties.

1. Quasiprojective varieties are "locally affine" (they are unions of affine varieties).
2. Quasiprojective varieties have a sheaf as part of their definition: $\mathcal{O}_X(X)$ (global sections), $\mathcal{O}_X(U)$ (regular functions), $\mathcal{O}_X, p$ (local rings, regular at a point p).
3. Schemes will be "locally affine" (unions of affine schemes) and also have a sheaf.
4. Quasiprojective varieties required an embedding in $\mathbb{P}^n$, whereas schemes are coordinate-free (intrinsically).
5. Affine varieties over $k \leftrightarrow$ finite generated k-algebras that are integral domains, whereas affine schemes $\leftrightarrow$ rings.

Definition 1. A presheaf $\mathcal{F}$ (of Abelian groups on a topological space X) consists of the following data: (1) to every $U \in \text{Top}(X)$, $\mathcal{F}$ assigns an Abelian group $\mathcal{F}(U)$. (2) to every inclusion $U \subset V$, $\mathcal{F}$ assigns a group homomorphism $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(Y)$ such that $\rho_{UU} = \text{id}$ and $U \subset V \subset W \Rightarrow \rho_{WU} = \rho_{VU} \circ \rho_{WV}$. (3) $\mathcal{F}(\varnothing) = 0$.

That is, a presheaf is a contravariant functor $\mathcal{F} : \text{Top}(X) \to Ab$. Note that $\mathcal{F}(X)$ are the global sections, $\mathcal{F}(U)$ are the sections on U, and ρ_{VU} is the restriction map. We denote $\rho_{VU}(f) = f\mid_U$.

Definition 2. A sheaf $\mathcal{F}$ is a presheaf satisfying (4, identity) if $\{U_i\}$ is an open cover for U and $f \in \mathcal{F}(U)$ such that $f\mid_{U_i}$ is 0 for all i, $f = 0$, and (5, gluing) if $\{U_i\}$ is an open cover for U, and if $\{f_i\} \subset \mathcal{F}(U_i)$ is a collection such that $f_i\mid_{U_i \cap U_j} = f_j\mid_{U_i \cap U_j}$ for all i, j, then there $f \in \mathcal{F}(U)$ such that $f\mid_{U_i} = f_i$.

Given the correct definitions, presheaves are common, but sheaves are useful.

Example.
- X a quasiprojective variety, $\mathcal{O}_X(U)$ the regular functions on U, for U Zariski open.
- X any topological space, $U \mapsto C(X, \mathbb{R})$.
- X smooth manifold, $U \mapsto C^\infty(X, \mathbb{R})$.
- (**Non-example**) $X = \mathbb{R}$, $U \mapsto BC(U, \mathbb{R})$ (gluing does not work for $(n, n+2)$ and $f(x) = x$).
- Sections of a Vector Bundle
  - X smooth manifold, $U \mapsto \Gamma^\infty(U)$, $s \in \Gamma^\infty(U)$ implies $s : U \to \sqcup_{p \in U} T_p X$ for s smooth. If we think of our tangent space as a line bundle, then the ideas of "sheaf" and "stalk" correlate nicely with their agricultural definitions.
  - X smooth manifold, $U \mapsto \Omega^k(U)$, the smooth differential k-forms on U ($s \in \Omega^k(U) \Rightarrow s : U \to \sqcup_{p \in U} \wedge^k (T_p X^*)$ where $s(p) \in \wedge^k (T_p X^*)$, s smooth.
- (**Non-example**) $X = \{0, 1\}$ with discrete topology, $\mathcal{F}(X) = \mathbb{R}^3$, restriction maps to first and second components, $\mathcal{F}(0) = \mathcal{F}(1) = \mathbb{R}$.

- (**Non-example**, constant presheaf) Fix Abelian group A, $\mathcal{F}(U) = A$ for all $U \neq \varnothing$. Restriction is the identity always, except to the empty set (of course). To see why this is a non-example: Let $X = \{0, 1\}$, $A = \mathbb{Z}$. Then $2 \in \mathcal{F}(0)$, $3 \in \mathcal{F}(1)$. Note $2 |_{0 \cap 1} = 3 |_{0\cap 1} = 0$. Hence there should exist an $n \in \mathbb{Z}$ such that $n |_0 = 2$ and $n |_1 = 3$. (Gluing is usually difficult over disconnected spaces.)

- X any topological space, A Abelian, $U \mapsto C(U, A)$ where A has the discrete topology. These functions here are locally constant (constant on any connected subset of U).

- Skyscraper Sheaf. X any topological space, A Abelian, $p \in X$ fixed. $\mathcal{F}(U) = i_{p,*}A(U) = \begin{cases} A \text{ if } p \in U \\ 0 \text{ otherwise} \end{cases}$.

Definition 3. For $p \in X$, $\mathcal{F}$ a presheaf, the <u>stalk</u> of $\mathcal{F}$ at p is $\mathcal{F}_p = \lim\limits_{\substack{\to \\ u \in \mathrm{Top}(X) \\ p \in U}} \mathcal{F}(U) = \{[f, U] \mid p \in U \in \mathrm{Top}(X), f \in \mathcal{F}(U), [f, U] = [g, V] \text{ if } \exists W, p \in W \subset U \cap V, \text{ s.t. } f |_W = g |_W \}$.

$[f, U] = [f |_W, W]$ for all $p \in W \subset U$. $\mathcal{F}_p$ is an Abelian group under the operation $[f, U] + [g, V] = [f |_{U\cap V} + g |_{U \cap V}, U \cap V]$, identity $[0, U]$. Notes: $[f, U] \in \mathcal{F}_p$ is called a <u>germ</u>. An example of this is $C_p^\infty(U, \mathbb{R})$, the set of functions defined on U smooth around p with the equivalence relation given in the definition above, and the group being that $\mathcal{O}_{x,p}$. The stalks of the constant presheaf (global), constant sheaf (local) are all A. For the skyscraper sheaf, the sheaves are A if in the closure of $\{p\}$ and 0 otherwise.

Definition 4. A <u>morphism</u> of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $\{\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)\}$ such that, if $U \subset V$, $\rho_{VU} \circ \phi_V = \phi_U \rho_{VU}$. That is, the maps commute with restriction maps, or equivalently $\phi_v(F) |_U = \phi_U(f |_U)$ (i.e., ϕ is a <u>natural transformation</u>). A morphism of sheaves is a morphism of presheaves. This morphism is an <u>isomorphism</u> if ϕ_U is an isomorphism for all $U \in \mathrm{Top}(X)$.

Proposition 1. *If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there is an induced group homomorphism on stalks $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ for all $p \in X$.*

Proof. Define $\phi_p([f, U]) = [\phi_U(f), U]$. Let us show this is well-defined. If $[f, U] = [g, V]$, there exists $W, p \in W \subset U \cap V$, such that $f |_W = g |_W$. So $\phi_w(f |_W) = \phi_W(g |_W) \Rightarrow \phi_U(f) |_W = \phi_V(g) |_W \Rightarrow [\phi_U(F), U] = [\phi_V(g), V]$ (this is by the equivalence relation inside the stalk) $\Rightarrow \phi_p([f, U]) = \phi_p([g, V])$. $\square$

Theorem 1. *Given ϕ is a morphism of sheaves, $\phi : \mathcal{F} \to \mathcal{G}$ is an isomorphism of sheaves iff the induced maps $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ are isomorphisms for all $p \in X$. That is, $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all U iff $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ an isomorphism for all p.*

Proof. ($\Rightarrow$) Assume $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all $U \in \mathrm{Top}(X)$. (Surjectivity) Let $[g, V] \in \mathcal{G}_p$. Then $g \in \mathcal{G}(V) \Rightarrow \exists f \in \mathcal{F}(V)$ such that $\phi_V(f) = g$. Then $\phi_p([f, V]) = [\phi_V(f), V] = [g, V]$. (Injectivity) Assume $\phi_p([f, U]) \overset{\text{Direct limit defn}}{=} [\phi_U(f), U] = 0 \in \mathcal{G}_p$. Then there exists $p \in W \subset U$ such that $f |_W = 0$. Then $\phi_W(f |_W) = 0 \Rightarrow \phi_U(f) |_W = 0 \Rightarrow \phi_W(f |_w) = 0 \Rightarrow f |_W = 0$ since ϕ_W is injective, so $[f, U] = 0 \in \mathcal{F}_p$.

($\Leftarrow$) Assume ϕ_p is an isomorphism for all $p \in X$. We will show ϕ_U is injective for any U. Assume $\phi_U(f) = 0 \Rightarrow [\phi_U(f), U] = [0, U] = 0 \in \mathcal{G}_p$ for all $p \in U$. So $\phi_p([f, U]) = [\phi_U(f), U] = 0 \in G_p$ for all $p \in X$. Hence $[f, U] = 0 \in \mathcal{F}_p$ for all $p \in X$. So for all $p \in X$, there exists $W_p, p \in W_p \subset U$ such that $f\ |_{W_p} = 0$. $\{W_p\}$ is an open cover for U, so $f = 0$.

Now we show ϕ_U is surjective for all U. Let $g \in \mathcal{G}(U)$. We want to show there is an $f \in \mathcal{F}(U)$ such that $\phi_U(f) = g$. Then $[g, U] \in G_p$ for all $p \in U$, so for all $p \in U$, there exists $f_p \in \mathcal{F}(V_p)$ such that $\phi_p([f_p, V_p]) = [\phi_{V_p}(f_p), V_p] = [g, U]$. Then for all $p \in U$, there is a $W_p, p \in W_p \subset V_p \cap U$ such that $\phi_{V_p}(f_p)\ |_{W_p} = g\ |_{W_p} \Rightarrow \phi_{W_p}(f_p\ |_{W_p}) = g\ |_{W_p}$ for all $p \in U$. Now for all $p, q \in U$, we get $\phi_{W_p}(f_p\ |_{W_p})\ |_{W_p \cap W_q} = (g\ |_{W_p})\ |_{W_p \cap W_q} \Rightarrow \phi_{W_p \cap W_q}(f_p\ |_{W_p \cap W_q}$ $) = g\ |_{W_p \cap W_q}$. Similarly, $\phi_{W_p \cap W_q}(f_q\ |_{W_p \cap W_q}) = g\ |_{W_p \cap W_q}$, and by injectivity of ϕ we get $f_p \cap_{W_p \cap W_q} = f_q\ |_{W_p \cap W_q}$. By gluing, then there exists $f \in F(U)$ such that $f\ |_{W_p} = f_p\ |_{W_p}$. Hence $\phi_u(f)\ |_{W_p} = \phi_{W_p}(f\ |_{W_p}) = \phi_{W_p}(f_p\ |_{W_p}) \overset{\text{From above}}{=} g\ |_{W_p}$ for all $p \in U$. These agree on every element of an open cover $\{W_p\}$. So $\phi_U(f) = g$. $\square$

Definition 5. Let $\mathcal{F}$ be a presheaf. The <u>sheafification of $\mathcal{F}$</u> $(\mathcal{F}^{\text{sh}}, \theta)$ is given by $\mathcal{F}^{\text{sh}}(U) = \{s : U \to \sqcup_{p \in U} \mathcal{F}_p : s(p) \in \mathcal{F}_p$ for all $p \in U$, and $\forall p \in U$, $\exists p \in V \subset U$ and $t \in \mathcal{F}(v)$ such that $s(q) = [t, V]\ \forall q \in V\}$, along with the sheafification morphism $\theta : \mathcal{F} \to \mathcal{F}^{\text{sh}}$ given by $\theta_u(f) : U \to \sqcup_{p \in U} \mathcal{F}_p$, $\theta_u(f)(p) = [f, U]$ for all $p \in U \in \text{Top}(X)$ and $f \in \mathcal{F}(U)$. Equivalently $\mathcal{F}^{\text{sh}}(U) = \{s : U \to \sqcup_{p \in U} \mathcal{F}_p \mid s(p) \in \mathcal{F}_p$ for all $p \in U$, and $\forall p \in U$, $\exists p \in V \subset U$ and $t \in \mathcal{F}(V)$ such that $s\ |_V = \theta_V(t)\}$. Note that $\sqcup_{p \in U} \mathcal{F}_p$ with a given topology is called Espace Étalé, so $\mathcal{F}^{\text{sh}}(U) = \{s : U \to \sqcup_{p \in U} \mathcal{F}_p \mid s(p) \in \mathcal{F}_p$ for all $p \in U$, and s is continuous with respect to this topology$\}$.

Theorem 2. *Let $\mathcal{F}$ be a presheaf, $(\mathcal{F}^{sh}, \theta)$ its sheafification. Then (1) $\mathcal{F}^{sh}$ is a sheaf, (2) $\mathcal{F}^{sh}_p \cong \mathcal{F}_p\ \forall p \in X$, (3) if $\mathcal{F}$ is a sheaf already, $\mathcal{F} \cong \mathcal{F}^{sh}$, (4) $(\mathcal{F}^{sh}, \theta)$ has the universal property; i.e., given $\phi : \mathcal{F} \to G$, there exists a unique $\Phi : \mathcal{F}^{sh} \to G$ that makes the obvious map commute.*

This θ map is a collection $\{\phi_u : \mathcal{F}(U) \to \mathcal{F}^{\text{sh}}(U)\}$ for each open U, which maps $f \mapsto \phi_U(f)$. Now this $\phi_U(f)$ is itself a function which maps $p \mapsto [f, U] \in \mathcal{F}_p$.

Proof. (of theorem) The fact that $\mathcal{F}^{\text{sh}}(U)$ is an Abelian group comes from the definition. Check that the morphisms work well, so that this definition indeed defines a presheaf. By definition of $\mathcal{F}^{\text{sh}}(U)$ as well, we get that this is a separated presheaf, so we must now show that the gluing property holds. Let $\{U_i\}$ be an open cover for U, $\{s_i \in \mathcal{F}^{\text{sh}}(U_i)\}$ satisfying $s_i\ |_{U_i \cap U_j} = s_j\ |_{U_i \cap U_j}$ for all i, j. Define $s \in \mathcal{F}^{\text{sh}}(U)$ by $s(p) = s_i(p)$ if $p \in U_i$ (this is well-defined by our requirement that s works well with overlaps in U_i). To determine that this s is indeed in $\mathcal{F}^{\text{sh}}(U)$, we check that $s(p) = s_i(p) \in \mathcal{F}_p$, and that $\forall p \in U$, the fact that $p \in U_i$ for some i implies that $\exists p \in V \subset U_i \subset U$ and $t \in \mathcal{F}(V)$ such that $\forall q \in V$, $s(q) = [t, V]$. This works by our definition of s and since $s_i \in \mathcal{F}^{\text{sh}}(U_i)$. This completes the proof of (1).

Now for (2), let $p \in X$ and define $\psi : \mathcal{F}^{\text{sh}}_p \to \mathcal{F}_p$ mapping $[s, U] \mapsto s(p)$. We claim this is well-defined. If $[s, U] = [t, V]$, then there exists $p \in W \subset U \cap V$ such that $s\ |_W = t\ |_W$. In particular, since $p \in W$, then $s(p) = t(p)$, so indeed this function is well-defined. For the homomorphism property, $\psi([s, U] + [t, V]) = \psi([s\ |_{U \cap V} + t\ |_{U \cap V}, U \cap V]) = (s\ |_{U \cap V}$ $+ t\ |_{U \cap V})(p) = s(p) + t(p) = (s + t)(p) = \psi([s, U]) + \psi([t, V])$. Injectivity is tricky, so we do surjectivity first. Let $[t, V] \in \mathcal{F}_p$. Then $t \in \mathcal{F}(V)$, so $\theta_V(t) \in \mathcal{F}^{\text{sh}}(V)$. Note $\psi([\theta_V(t), V]) = \theta_V(t)(p) \overset{\text{Defn of } \theta}{=} [t, V]$. For injectivity, assume $\psi([s, U]) = \bar{0} \in \mathcal{F}_p$, which implies that $s(p) = \bar{0} = \mathcal{F}_p$. Since $s \in \mathcal{F}^{\text{sh}}(U)$, there exists $p \in V_p \subset U$ and $t_p \in \mathcal{F}(V_p)$ such that $s\mid_{V_p} = \theta_{V_p}(t_p)$ (this is by the second property in the "equivalent" definition of $\mathcal{F}^{\text{sh}}(U)$. In particular, $0 = s(p) = [t_p, V_p] \Rightarrow \exists p \in W_p \subset V_p \subset U$ such that $t_p\mid_{W_p} = 0$ (this is by the second proeprty in the original definition of $\mathcal{F}^{\text{sh}}(U)$) and $s\mid_{V_p} = \theta_{V_p}(t_p) \Rightarrow (s\mid_{V_p})\mid_{W_p} = \theta_{V_p}(t_p)\mid_{W_p} \Rightarrow s\mid_{W_p} = \theta_{W_p}(t_p\mid_{W_p}) = \theta_{W_p}(0) = 0 \Rightarrow [s, U] = 0 \in \mathcal{F}_p^{\text{sh}}$. This gives us our isomorphism and completes the proof of (2).

Recall the theorem that states that a morphism of sheaves is an isomorphism iff the induced maps $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ where $\phi_p([f, U]) = [\phi_U(f), U]$ are isomorphisms for all $p \in X$. We will attempt to show these induced maps are indeed isomorphisms so that we can get the desired isomorphism for (3). Recall that $\theta_u : \mathcal{F}(U) \to \mathcal{F}^{\text{sh}}(U)$ maps $\theta_u(F)(p) = [f, U]$. So $\theta_p : \mathcal{F}_p \to \mathcal{F}_p^{\text{sh}}$ where $\theta_p([f, U]) = [\theta_u(f), U] \overset{\psi}{\leftrightarrow} \theta_u(f)(p) = [f, U]$. Hence θ_p is the identity map for all p, and by the one-to-one correspondence this is an isomorphism.

(4) We will prove this later. ☐

We give some definitions. $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Then ϕ is <u>injective</u> if $\ker\phi = 0$ and ϕ is <u>surjective</u> if $\operatorname{Im}\phi := \operatorname{Im}_{\text{pre}} \phi^{\text{sh}} \cong G$. We say $\mathcal{F}^{\trianglelefteq} \leq \mathcal{F}$, or that $\mathcal{F}^{\trianglelefteq}$ is a <u>subsheaf</u> of $\mathcal{F}$, if $\mathcal{F}^{\trianglelefteq}(U) \leq \mathcal{F}(U)$ for all U. A consequence of this is that $\mathcal{F}_p^{\trianglelefteq} \leq \mathcal{F}_p$ for all p. $\mathcal{F}^{\trianglelefteq}$ is also considered a subsheaf of $\mathcal{F}$ is there is an injective morphism $i : \mathcal{F}^{\trianglelefteq} \to \mathcal{F}$. Take $\mathcal{F}/\mathcal{F}^{\trianglelefteq}_{\text{pre}}(U) = \mathcal{F}(U)/\mathcal{F}^{\trianglelefteq}(U)$; then we get a <u>quotient sheaf</u> by sheafification (!). Note that $\ker\phi \leq F$ and $\operatorname{Im}\phi \leq G$.

Proposition 2. *$\phi : \mathcal{F} \to G$ is as above. Then ϕ is an isomorphism iff ϕ_U is an isomorphism $\forall U$ iff ϕ_p if an isomorphism $\forall p$ (by definition and by previous proposition). Furthermore, ϕ is injective $\iff$ ϕ_U is injective $\forall U$ iff ϕ_p is injective $\forall p$, and ϕ is surjective $\iff$ ϕ_p is surjective $\forall p$ (we don't get that ϕ surjective implies that ϕ_U is surjective for all U; see the proof of the previous proposition). If ϕ_U is surjective for all U, then ϕ and ϕ_p are surjective.*

In the last statement: that ϕ_U is surjective for all U implies that ϕ_p is surjective for all p is clear by definition, but the reverse takes some work.

(10.10)

Proposition 3. *If $\mathcal{F}$ satisfies the identity axiom (i.e., $\mathcal{F}$ is a separated presheaf), then θ is injective (so in this case $\mathcal{F} \leq \mathcal{F}^{sh}$, and $\mathcal{F}$ is a <u>subpresheaf</u>).*

Proof. Recall that $\theta_U : \mathcal{F}(U) \to \mathcal{F}^{\text{sh}}(U)$ is given by $\theta_U(f)(p) = [f, U]$ (this is by definition of what the sheafification of a presheaf is). If $\theta_U(f) = 0$, then $\theta_U(f)(p) = [f, U] = [0, U] = \bar{0} \in \mathcal{F}_p$ for all p. Hence $\forall p$, $\exists p \in W_p \subset U$ such that $f\mid_{W_p} = 0$. Now $\{W_p\}$ is an open cover for U, so $f = 0$ by the identity property. ☐

Theorem 3. *Let $\mathcal{F}$ be a presheaf, $(\mathcal{F}^{sh}, \theta)$ its sheafification. Then $(\mathcal{F}^{sh}, \theta)$ satisfies the following universal property: If G is a sheaf and $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism, then $\exists!$ morphism $\Phi : \mathcal{F}^{sh} \to \mathcal{G}$ such that $\Phi \circ \theta = \phi$.*

Proof. We define $\Phi_U : \mathcal{F}^{\text{sh}}(U) \to \mathcal{G}^{\text{sh}}(U) \cong \mathcal{G}(U)$ where $\Phi_U(s) : U \to \sqcup_{p\in U}\mathcal{G}_p$ is defined by $p \mapsto \Phi_U(s)(p) = \phi_p(s(p))$ (again, recall s is a function in $\mathcal{F}^{\text{sh}}(U)$, so $s(p)$ makes sense).

We want to show this is well-defined. First we show that $\Phi_U(s) \in \mathcal{G}^{\text{sh}}(U)$ for all $U \in \operatorname{Top}(X)$, $s \in \mathcal{F}^{\text{sh}}(U)$. We note that $\Phi_U(s)(p) \in \mathcal{G}_p$ for all p by our definition of it, as per condition (1) for sheafification. Furthermore, (we will choose $\tilde{t} = \phi_V(t)$, but our definitions of V, t come later), then for all $p \in U$, there $\exists p \in V \subset U$ and $\tilde{t} \in G(V)$ such that $\forall q \in V$, $\Phi_U(s)(q) = [\tilde{t}, V]$. This is since $s \in \mathcal{F}^{\text{sh}}(U)$, so that for all $p \in U$, $\exists p \in V \subset U$ and $t \in F(V)$ such that $\forall q \in V$, $s(q) = [t, V]$. Then $\Phi_U(s)(q) = \phi_q(s(q)) = \phi_q([t, V]) = [\phi_V(t), V]$ for all $q \in V$, proving (2) in sheafification. We also can show that Φ_U is a group homomorphism, and that if $W \subset U$, $\Phi_U(s) \mid_W = \Phi_W(s \mid_W)$. This will show that we have a well-defined morphism.

Now we outline how $\Phi \circ \theta = \phi$. We recall the lemma from before that, if $\psi, \psi^{\trianglelefteq} : \mathcal{F} \to \mathcal{G}$ are morphisms of sheaves, then $\psi = \psi^{\trianglelefteq} \iff \psi_p = \psi_p^{\trianglelefteq}\ \forall p \in X$, the induced maps. We can then show that $(\Phi \circ \theta)_p = \Theta_p \circ \theta_p = \phi_p$ for all p which are simple and left to the reader as an exercise. :)

Finally we show that Φ is unique. If $\exists \Psi : \mathcal{F}^{\text{sh}} \to \mathcal{G}^{\text{sh}}$ such that $\Psi \circ \theta = \phi = \Phi \circ \theta$, then this implies $\Psi_p \circ \theta_p = \Phi_p \circ \theta_p$. Now since θ_p is an isomorphism (as we have seen in a previous lemma), we get that $\Psi_p = \Phi_p\ \forall p$, which implies $\Psi = \Phi$, giving uniqueness. $\square$

Definition 6. If $f : X \to Y$ is continuous and $\mathcal{F}$ is a sheaf on X, we define the direct image sheaf $(f_*\mathcal{F})(V)$ to be a sheaf on Y where $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ where restrictions come from $\mathcal{F}$. If $f : X \to Y$ is continuous and $\mathcal{G}$ is a sheaf, then the inverse image sheaf $f^{-1}\mathcal{G}$ is a sheaf on X where for all $U \in \text{Top}(X)$, $f^{-1}\mathcal{G}(U) = \varinjlim_{\substack{V \in \text{Top}(Y) \\ f(U) \subset V}} \mathcal{G}(V)$; this sheaf is given by sheafifying the resulting presheaf from this construction.

Example. Let $i_p : \{p\} \hookrightarrow X$ for $p \in X$. Let $\mathcal{F}$ be the constant sheaf $\underline{A}$ on $\{p\}$ where A is an Abelian group (recall: this sends $\{p\}$ to A and $\varnothing$ to 0). Then $i_{p,*}\underline{A}$ is the direct image sheaf such that, $\forall V \in \text{Top}(X)$, $i_{p,*}\underline{A}(V) = \underline{A}(i_p^{-1}(V)) = \begin{cases} A \text{ if } p \in V \\ 0 \text{ if } p \notin V \end{cases}$. This is deemed the skyscraper sheaf; we have seen this before.

Example. Let $i : Z \hookrightarrow X$ be inclusion, where $\mathcal{G}$ is a sheaf on X. Then $\forall U \cap Z \in \text{Top}(Z)$, $i^{-1}\mathcal{G}(U \cap Z) = \varinjlim_{U \cap Z \subset V} \mathcal{G}(V)$ (sheafify). In the special case where Z is open in X, $i^{-1}\mathcal{G}(U \cap Z) = \mathcal{G}(U \cap Z)$, so this does not need sheafification. In this case, we denote $i^{-1}\mathcal{G}$ as $\mathcal{G} \mid_Z$, a sheaf on Z.

Here is a note: we have $(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)}$ for all $p \in X$. On the other hand, f_* and f^{-1} are adjoint functors; that is, $\text{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \text{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$.

2.2 Affine Schemes. I am ready for more categories.

Definition 7. A ringed space $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ called the structure sheaf of X (on the topology of X).

A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that the stalks $\mathcal{O}_{X,p}$ are local rings.

A morphism of ringed spaces $(f, f^\# : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ (or equivalent a morphism of sheaves $f^\# : \mathcal{O}_X \to f^{-1}\mathcal{O}_Y$). Recall that $f_*(\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$ for all $V \in \mathrm{Top}(Y)$, and since $f^{-1}(V)$ is open since f is continuous, this makes sense.

A local homomorphism is a ring homomorphism $\phi : A \to B$ of local rings such that $\phi^{-1}(m_B) = m_A$ where m_A, m_B are maximal ideals of their respective spaces.

A morphism of locally ringed spaces is a morhpism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that the induced maps on stalks $f_p^\# : \mathcal{O}_{Y,p} \to f_*\mathcal{O}_{X,p}$ is a local homomorphism $\forall p \in X$.

An isomorphism of locally ringed spaces is a morphism where f is a homeomorphism and $f^\#$ is an isomorphism of sheaves. If this $(f, f^\#)$ exists, we say $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are isomorphic.

A manifold of dimension n is a locally ringed space (X, θ_X) where X is second countable Hausdorff and X has a cover $\{U_i\}$ such that $(U_i, \mathcal{O}_X \mid_{U_i})$ is isomorphic (as locally ringed spaces) to (U, C_U^o) for some open $U \subset \mathbb{R}^n$. Here C_U^o is defined to be the continuous functions $C^0(V, \mathbb{R})$ for all $V \subset U$ open.

Example. Let M be a smooth n-manifold. Then we can define $T_pM := \{\text{linear } D : C^\infty(\mathbb{R}) \to \mathbb{C}^\infty(\mathbb{R}) \mid D(fg) = D(f)g(p) + f(p)D(g)\} = \{\text{linear } D : C_p^\infty(\mathbb{R}) \to \mathbb{C}_p^\infty(\mathbb{R}) \mid \text{ ditto }\}$. We are heavily using that the functions $C^\infty(\mathbb{R})$ on $\mathbb{R}$ form a sheaf, so that we can retrieve global information from the local information given from stalks. This is equal to $\{\text{linear } D : m_p \to C_p^\infty(\mathbb{R})/m_p \cong \mathbb{R}\}$ where $m_p = \{f \in C_p^\infty(\mathbb{R}) \mid f(p) = 0 \mid \text{ditto }\}$, since we might as well only consider derivatives of functions whose values at $p = 0$ since derivatives are translation-invariant. This equals $\{$ linear $D : m_p/m_p^2 \to \mathbb{R}\}$, the definition of tangent space as given, where $m_p/m_p^2 = T_pX^*$, the cotangent space.

Definition 8. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules (or $\mathcal{O}_X$-module) is a sheaf of Abelian groups $\mathcal{F}$ on X such that $\forall U \in \mathrm{Top}(X)$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and where $\rho_{WU} : \mathcal{F}(U) \to \mathcal{F}(W)$ (always a group homomorphism) is an $\mathcal{O}_X(U)$-module homomorphism, where $\mathcal{F}(W)$ is given the $\mathcal{O}_X(U)$-module structure from the pushforward of $\rho_{WU} : \theta_X(U) \to \theta_X(W)$ (a ring homomorphism). Once again, if $\phi : A \to B$ is a group homomorphism, then B is an A-module where $a.b = \phi(a)b$; this is the pushforward.

Our goal is to associate to every ring A a locally ringed space (an affine scheme) $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$; to every A-module M, an $\mathcal{O}_{\operatorname{Spec}(A)}$-module $\tilde{M}$ (a quasicoherent sheaf); and if A is Nötherian and M is finitely generated, an $\mathcal{O}_{\operatorname{Spec}(A)}$-module $\tilde{M}$ (a coherent sheaf).

We return to varieties for the moment. Recall that a morphism of quasiprojective varieties is a continuous $\phi : X \to Y$ such that, for every regular $f \in \mathcal{O}_Y(U)$, $f \circ \phi$ is regular as an element in $\mathcal{O}_X(f^{-1}(U)) = \phi_*\mathcal{O}_X(U)$. Hence with such a morphism we may define a map $\phi^\# : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ given $\phi_U^\# : \mathcal{O}_Y(U) \to \phi_*\mathcal{O}_X(U) = \mathcal{O}_X(\phi^{-1}(U))$ mapping $f \mapsto f \circ \phi$. This gives an example of the pushforward structure given here and relates our definition of regular to our definitions.

Definition 9. An (abstract) algebraic variety is a locally ringed space $(X, \mathcal{O}_X)$ with a cover $\{U_i\}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine variety $\forall i$.

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ with a cover $\{U_i\}$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to $(\mathrm{Spec}(A_i), \mathcal{O}_{\mathrm{Spec}(A_i)})$ for some rings A_i for all i.

Although we have not discussed all the following inclusions, we have that Ringed spaces$\supset$ Locally ringed spaces$\supset$ Manifolds, and Locally ringed spaces $\supset$ Schemes $\supset$ Varieties.

(10.24)

Definition 10. Let R be a ring. The (prime) spectrum of R, denoted $\mathrm{Spec}\, R$, is the set of all prime ideals of A.

We note that elements of $\mathrm{Spec}(R)$ are denoted $[P]$ for $P \trianglelefteq R$ prime. We define V : $\{\text{Ideals of } R\} \to \{\text{Subsets of Spec}(R)\}$ such that $V(J) = \{[P] \in \mathrm{Spec}(R) \mid f([P]) = 0\, \forall f \in J\}$, where $f([P]) := f(\text{mod } p)$ in R/P. This notation comes from thinking of elements in our ring R as *functions*: we can write $f \in R \Rightarrow f : \mathrm{Spec}(R) \to \sqcup_{[P]\in\mathrm{Spec}(R)} R/P$, where $f([P]) \in R/P$.

Definition 11. $V(I) := \{[P] \in \mathrm{Spec}(R) \mid P \supset I\}$. The Zariski topology on $\mathrm{Spec}(R)$ has as closed subsets $V(I)$ for all ideals $I \trianglelefteq R$.

We prove this is a topology. We get $V((0)) = \mathrm{Spec}(R)$, and $V(R) = \varnothing$ since we ask that our prime ideals be proper. $V(J) \cup V(K) = V(JK)$ since J, K are prime. $\bigcap_i V(J_i) = V(\sum_i J_i)$. Since $JK, \sum_i J_i$ are prime ideals, we are done.

Here's where we will end up: we will be defining a sheaf of rings where our set is $\mathrm{Spec}(R)$ with the Zariski topology and the global sections $\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R) \cong R$.

If $\theta : R \to S$ is a ring homomorphism, we can define a continuous map $\mathrm{Spec}(\theta)$: $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ by $\mathrm{Spec}(\theta)([P]) = [\theta^{-1}(P)]$. The fact that such a function exists comes from the correspondence theorem for (prime) ideals. For proof of continuity in the Zariski topology, we observe that $\mathrm{Spec}(\theta)^{-1}(V(J)) = V(\theta(J))$, so closed sets pull back to closed sets.

We make some notes here: if $\phi : R \to S$ is injective, $\mathrm{Spec}(R)$ is a quotient space of $\mathrm{Spec}(S)$. If $\phi : R \to S$ is surjective, then we can embed $\mathrm{Spec}(S) \subset \mathrm{Spec}(R)$ (as a subspace).

Let us observe how this Zariski topology behaves on $\mathrm{Spec}(R)$. Given $[P] \in \mathrm{Spec}(R)$, we get $\overline{[P]} = \bigcap_{C \supset [P] \text{ closed}} = C = \bigcap_{[P]\in V(I))} V(I) = \bigcap_{I \subset P} V(I) = V(P)$. That is, the closure of a point is itself in $\mathrm{Spec}(R)$ iff P is maximal.

We would like to get a version of Hilbert's Nullstellensatz for this topology. For proving this in the case of varieties, we needed that the maximal ideals of $k[x_1, \ldots, x_n]$ were all of the form $(x_1 - a_1), \ldots, (x_n - a_n)$ for $a_i \in k$; plus we needed a Rabinowitsch trick. We claim that the analogs of these statements for sheafs are trivial.

We quickly recall that $V(J) := \{p \in A^n(k) \mid f(p) = 0\, \forall f \in J\}$ and that $I(W) := \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0\, \forall p \in W\}$ where V maps from ideals of $k[x_1, \ldots, x_n]$ to subset of $A^n(k)$ and I maps from subset of $A^n(k)$ to ideals of $k[x_1, \ldots, x_n]$ and $A^n(k)$ is n-affine space. The weak Nullstellensatz for sheafs says that, if $I \subsetneq R$ is a proper ideal, $V(I) \neq \varnothing$. Every proper ideal is contained in a maximal ideal, so this comes directly. For the strong version, define $I(W) := \{f \in R \mid f([P]) = 0\, \forall [P] \in W\} = \{f \in R \mid f \in P\, \forall [P] \in W\} = \bigcap_{[P]\in W} P$ for all $W \in \mathrm{Spec}(R)$. But then we can prove what is desired, since $I(V(J)) = \bigcap_{[P]\in V(J)} P = \bigcap_{P \supset J} P =: \mathrm{Rad}(J)$ by the well-known theorem that the radical of an ideal is the intersection of the prime ideals containing it.

Having this information, we look at some examples of affine schemes. Recall that, given different fields, $V(x^2+1)$ looks different: if $\mathbb{R}$ is our field this space is empty, but $V(x^2+1)$ is (x^2+1) when our field over our affine space is $\mathbb{C}$. This leads into our first example:

Example. Spec(Field). Looking at Spec($\mathbb{R}$) and Spec($\mathbb{C}$), these two are fundamentally different even at $[(0)]$. The global sections on this point in Spec($\mathbb{R}$) are isomorphic to $\mathbb{R}$ (requires justification), whereas on Spec($\mathbb{C}$) these global sections are isomorphic to $\mathbb{C}$, and $\mathbb{R}$, $\mathbb{C}$ are not isomorphic.

Spec(DVR), where DVR is discrete valuation ring, meaning PIDs with exactly one non-zero prime ideal. We start with a two-point space $[(0)]$ and $[(P)]$. Our topology is the empty set, $\{[(0)]\}$, and Spec(R), since the Zariski topology on this space says the only closed space is $[(P)]$.

Spec($\mathbb{Z}$). The prime ideals are (p) where p is prime. $[(0)]$ is considered a <u>fat point</u>, since (topologically speaking) it lives inside every other point.

Spec($\mathbb{C}[x]$) $=: \mathbb{A}^1(\mathbb{C})$. The prime ideals are $(x-a)$ or (0), the latter again being a fat point. The geometric picture for this is $\mathbb{C} \cong \mathbb{R}^2$.

Spec($\mathbb{R}[x]$) $= \{[(0)], [(x-a)](a \in \mathbb{R}), [(ax^2+bx+c)](a, b, c \in \mathbb{R}, b^2-4ac < 0)\}$. These are all the prime ideals in $\mathbb{R}[x]$ since any polynomial can be factored into linears and quadratics (where coefficients of all polynomials are in $\mathbb{R}$). Geometrically speaking, we can think of Spec($\mathbb{R}[x]$) as a quotient of Spec($\mathbb{C}[x]$) in the following way: every linear $(x-a)$ can be mapped to the horizontal axis by the correspondence $a \leftrightarrow (x-a)$. For quadratics, we can map (ax^2+bx+c) to the zero with positive imaginary part. This is a one-to-one correspondence, so Spec($\mathbb{R}[x]$) can be thought of as a the closed upper half-plane.

(Skipped 10.31; see Byeongsu's notes.)

(11.7) Recall that we redefined our affine spaces $\mathbb{A}^2_{\mathbb{C}}$ and $\mathbb{A}^3_{\mathbb{C}}$ to be Spec($\mathbb{C}[x, y]$) and Spec($\mathbb{C}, [x, y, z]$), respectively. This space the fat point $[(0)]$, the less fat points (f) for $f \in \mathbb{C}(x, y)$ (whose closures correspond to the one-dimensional vanishing sets typical in $\mathbb{A}^2_{\mathbb{C}}$), and the maximal ideals $[(x-a, y-b)]$ (which correspond to single points in the plane). We have mentioned that the closures are indeed these vanishing sets, and to see this we note that $\bar{S} = V(I(S))$ (which will be <u>proven</u> in the morning sessions) where V is vanishing of an ideal and $I(S) = \cap_{[P] \in S} P$. Hence $\overline{\{[(x)]\}} = V(I(x)) = V(x) = \{[(x)]\} \cup \{[(x, y-b)] \mid b \in \mathbb{C}\}$. So the closure is indeed this fat point unioned with the y-axis. Also, $\overline{\{[(x^2+y^2-1)]\}} = V(I(x^2+y^2-1)) = V(x^2+y^2-1) = \{[(x-a, y-b)] \mid a^2+b^2-1 = 0\} \cup \{[(x^2+y^2-1)]\}$.

Now in $\mathbb{A}^3_{\mathbb{C}}$, the ideals corresponding to surfaces are not so simple, as can be seen by the example of the twisted cubic which is the closure of $(xz-y^2, y-z^2, x-yz)$. There are still lines generated by just one element - the x-axis is the closure of the fat point by (y, z). Note that the closure of $[(z)]$ contains the entire xy-plane as well as any *curve* living on the xy-plane.

Now we move on to the spectrums of quotient rings. For any ideal J, we have Spec(R/J) $\leftrightarrow \{[P] \in \text{Spec}(R) \mid P \supset J\} \hookrightarrow$ Spec(R) where the embedding is Spec(ϕ) : Spec(R/J) $\to$ Spec(R) is given by $[P] \mapsto [\phi^{-1}(P)]$ where ϕ is the canonical projection of R onto R/J.

Example. Spec($\mathbb{C}[x, y]/(y-x^2)$) is the parabola $y = x^2$ together with the fat point $[(0)]$, or the vanishing set of $(y-x^2)$ with this fat point. Also, Spec($\mathbb{C}[x, y]/(xy)$) is the coordinate axes with the fat points $(x), (y)$, or the vanishing set of (xy) with these two fat points.

Also, $\operatorname{Spec}(\mathbb{C}[x,y,z]/(x^2+y^2+z^2-1))$ is the unit sphere in $\mathbb{A}^3_{\mathbb{C}}$, together with all fat points whose closures are curves in $\mathbb{A}^3_{\mathbb{C}}$ intersected with the sphere (the intersection of prime ideals is prime).

Let $S \subset R$ is a multiplicative subset. Then $\operatorname{Spec}(S^{-1}R) \leftrightarrow \{[P]\operatorname{Spec}(R) \mid P \cap S = \varnothing\} \subset \operatorname{Spec}(R)$ (not necessarily an embedding here if R is not an integral domain; the canonical projection $\phi : R \to S^{-1}R$ is not necessarily injective).

There are two flavors of these localizations, given by different designations of the multiplicative subset: (1) $S = R \setminus P, P \trianglelefteq R$, which yields R_p; (2) $S = \{1, f, f^2, \dots\}$ for $f \in R$, which yields R_f. Note that (1) $\operatorname{Spec}(R_p) \leftrightarrow \{[Q] \in \operatorname{Spec}(R) \mid Q \cap (R \setminus P) = \varnothing\} = \{[Q] \in \operatorname{Spec}(R) \mid Q \subset P\}$; and (2) $\operatorname{Spec}(R_f) \leftrightarrow \{[P] \in \operatorname{Spec}(R) \mid P \cap \{1, f, f^2, \dots\} = \varnothing\} \overset{P \text{ prime}}{=} \{[P] \in \operatorname{Spec}(R) \mid P \cap \{f\} = \varnothing\} = \{[P] \in \operatorname{Spec}(R) \mid f \notin P\} = \operatorname{Spec}(R) \setminus V(f) =: D(f)$, a distinguished open set, since $V(f) := V((f)) = \{[P] \in \operatorname{Spec}(R) \mid f \in P\}$ by definition of V.

Example. (Of (1)) $\operatorname{Spec}(\mathbb{C}[x,y]_{(x,y)})$. The only closed point is $[(x,y)]$ which exists at the origin of our picture, and the only other (fat) points that exist are $[(0)]$ and those that correspond to points going through the origin $[(x,y)]$; this is since we have localized at this point. For instance, $(y - x^2)$ is contained within (x,y), so the closure of $(y - x^2)$ is a parabola that goes through this origin, and $(y - x^2)$ is in our spectrum. However, the unit circle $(x^2 + y^2 - 1)$ is *not* in this picture.

$\operatorname{Spec}(\mathbb{C}[x,y,z]_{(y,z)})$. This has the fat points (0), (y,z), (z), (y), and all those curves in y and z going through the x-axis. For instance, $(y - z^2)$.

(Of (2)) $\operatorname{Spec}(\mathbb{C}[x,y]_x)$. This is the affine space minus the x-axis and the fat point (x), as can be seen by our exposition of (2) above. Also, $\operatorname{Spec}(\mathbb{C}[x,y,z]_{x^2+y^2-z^2})$ is the entire affine 3-space minus the cone.

(11.14) We now discuss function on $\operatorname{Spec}(R)$: for all $f \in R$, we may make f a function on $\operatorname{Spec}(R)$ into $\sqcup_{[P]\in\operatorname{Spec}(R)} R/P \hookrightarrow \sqcup_{[P]\in\operatorname{Spec}(R)} k_P(x)$ where $[P] \mapsto f([P]) := f \mod P$. For instance, $(x^3 - 27) \mod (x-2)$ is 35, and $(x^3 - 27) \mod (x - i)$ is $27 - i$. We can generalize the topology on vector/fiber bundles to put a topology on $\operatorname{Spec}(R)$ that is the weakest one such that all such maps f are continuous. (This will be formalized at morning meetings.) We ask: what $\frac{f}{g}$ are defined at $[P] \in \operatorname{Spec}(R)$? The answer: if $g([P]) \neq 0$; i.e., $g \notin P$. In other words, $\frac{f}{g} \in R_P$ (localization), which is $(R/P)^{-1}R$. We ask: what is the domain of some $\frac{f}{g}$? The answer: $\{[P] \in \operatorname{Spec}(R) \mid g([P]) \neq 0\} = \{[P] \in \operatorname{Spec}(R) \mid g \notin P\} = \{[P] \in \operatorname{Spec}(R) \mid P \not\supset (g)\} = \operatorname{Spec}(R) \setminus V(g) =: D(g)$, called the <u>distinguished open set</u>. (3) What $\frac{f}{g}$ are defined on $D(h)$? Answer: we cannot have any elements in the denominator that might vanish on $D(h)$, so the only elements that will suffice will be powers of h itself. Hence the set we search for is $\{\frac{a}{h^k} : a \in R, k \in \mathbb{N}\} = R_h = \{1, h, h^2, \dots\}^{-1}R$.

We claim that $\{D(f) : f \in R\}$ forms a basis for the Zariski topology on $\operatorname{Spec}(R)$. These distinguished open sets do not account for *every* open set in this topology. Note that $D(x) \subset \operatorname{Spec}(\mathbb{C}[x,y])$ is everything but the y-axis, and $D(y) \subset \operatorname{Spec}(\mathbb{C}[x,y])$ is everything but the x-axis. Hence $\operatorname{Spec}(R) \setminus V(x,y) = D(x) \cup D(y)$ is not a distinguished affine open set in itself but will be open.

Definition 12. Let X be a topological space with basis $\mathcal{B} = \{B_i\}_{i\in I}$. A <u>$\mathcal{B}$-sheaf</u> $\mathcal{F}$ on X consists of the following data: (1) $\mathcal{F}$ assigns to each B_i an Abelian group $\mathcal{F}(B_i)$; (2) $\mathcal{F}$ assigns to every inclusion $B_j \subset B_i$ in $\mathcal{B}$ a group homomorphism $\rho_{ij} : \mathcal{F}(B_i) \to \mathcal{F}(B_j)$

satisfying (a) $\rho_{ii} = \iota_{\mathcal{F}(B_i)}$ and (b) $B_k \subset B_j \subset B_i$ implies $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$; (3) if $B \in \mathcal{B}$ and $\{B_i\}_{i \in J \subset I}$ is a cover for B with $f \in \mathcal{F}(B)$ satisfying $f \mid_{B_i} = 0$ in $\mathcal{F}(B_i)$ for all $i \in J$, then $f = 0$; (4) if $B \in \mathcal{B}$ and $\{B_i\}_{i \in J \subset I}$ is a cover for B with $\{f_i \in \mathcal{F}(B_i)\}$ a collection satisfying $f_i \mid_{B_k} = f_j \mid_{B_k}$ for all $i, j \in J$ and $B_k \in \mathcal{B}$ such that $B_k \subset B_i \cap B_j$, then there exists $f \in \mathcal{F}(B)$ such that $f \mid_{B_i} = f_i$ for all $i \in J$.

Definition 13. Let $\mathcal{F}$ be a $\mathcal{B}$-sheaf on X. Then $\forall p \in X$, the stalk at $\mathcal{F}$ at p is $\mathcal{F}_p = \lim_{\overrightarrow{p \in B_i \in \mathcal{B}}} \mathcal{F}(B_i) = \{[f, B_i] \mid B_i \in \mathcal{B}, f \in \mathcal{F}(B_i)$, and $[f, B_i] = [g, B_j]$ if $\exists B_k \in \mathcal{B}$ such that $p \in B_k \subset B_i \cap B_j$ and $f \mid_{B_k} = g \mid_{B_k}$.

Definition 14. Let $\mathcal{F}$ be a $\mathcal{B}$-sheaf on X. The sheafification of $\mathcal{F}$ is given by the data: $\forall U \in \mathrm{Top}(X)$, $\mathcal{F}^{\mathrm{sh}}(U) = \{s : U \to \sqcup_{p \in U} \mathcal{F}_p \mid (1) s(p) \in \mathcal{F}_p \, \forall p \in U, \quad (2) \forall p \in U, \exists p \in B_i \in \mathcal{B}$ with $B_i \subset U$ and $f \in \mathcal{F}(B_i)$ such that $\forall q \in B_i$, $s(q) = [f, B_i]$ in $\mathcal{F}_q\}$ and with restriction maps being as usual.

Proposition. *$\mathcal{F}^{sh}$ is a sheaf on X. For all $p \in X$, $\mathcal{F}^{sh}_p \cong \mathcal{F}_p$. For all $B_i \in \mathcal{B}$, $\mathcal{F}^{sh}(B_i) \cong \mathcal{F}(B_i)$.*

Proofs here are similar to those given before. In (2) we use the map $\phi : \mathcal{F}^{\mathrm{sh}}_p \to \mathcal{F}_p$ given by $\phi([s, U]) = s(p) \in \mathcal{F}_p$, and in (3) we use $\psi : \mathcal{F}(B_i) \to \mathcal{F}^{\mathrm{sh}}(B_i)$ mapping $f \mapsto \tilde{f}$ where $\tilde{f}(p) = [f, B_i]$ for all $p \in B_i$.

We now wish to extend our functions on $\mathrm{Spec}(R)$ in order to make a sheaf. Let $\mathcal{B} = \{D(f) \mid f \in R\}$ be our basis. Let $\mathcal{F}(D(f)) = R_f$ be our $\mathcal{B}$-sheaf on $\mathrm{Spec}(R)$, with the following restriction maps: if $D(f) \subset D(g)$, then $f \in \mathrm{Rad}(g)$ (Hilbert), so $f^k = gh$ for some $k \in N$ and $h \in R$. We may define $\rho_{gf} : R_g \to R_f$ by mapping $\frac{a}{g^m} \mapsto \frac{ah^m}{g^m h^m} = \frac{ah^m}{f^{km}} \in R_f$. (This is a ring homomorphism. Believe it.) We claim that $\mathcal{O}_{\mathrm{Spec}(R),[p]} \cong R_p$. Define the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ to be the sheafification of our $\mathcal{B}$-sheaf. Then for all $U \in \mathrm{Top}(X)$, $\mathcal{O}_{\mathrm{Spec}(R)}(U) = \{s : U \to \sqcup_{[P] \in U} R_p \mid (1) s([P]) \in R_p \, \forall [P] \in U \quad (2) \forall [P] \in U, \exists [P] \in D(f) \subset U$ and $\frac{a}{f^k} \in R_f$ such that $\forall [a] \in D(f)$, $s([Q]) = \frac{a}{f^k}$ in $R_q\}$.

Theorem. *(1) $\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \cong R$. (2) $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) \cong R_f$. (3) $\mathcal{O}_{\mathrm{Spec}(R),[P]} \cong R_p$.*

Proof. For (1), note that $\mathrm{Spec}(R) = D(1)$. The rest come directly from the previous proposition that the stalks and (local) rings associated with $B \in \mathcal{B}$ in the $\mathcal{B}$-sheaf are the isomorphic to those in the sheafification. $\square$