Basic Algebraic Geometry I

Notes by Byeongsu Yu

November 5, 2018

Abstract

This note is based on the SWAG(Student Working Algebraic Geomtery) seminar's presentation, Fall 2018 at Texas A&M University, based on Algebraic Geometry by Hartshorne. Much part of this note was TEX-ed after seminar.

Contents

1 Sep 12, 2018: Chapter 2.1

Lectured by C. J. Bott,

1.1 Motivation

To study schemes, an important and fun object. We assume that

- k : algebraically closed field. ($k^x = k \setminus \{0\}$, multiplicative group of field.)
- Every ring is commutative ring with unity.
- X be topological space
- $Top(X) = \{$ open subsets of $X\}$.

Recall definitions from Chapter 1.

- Projective space

$$\mathbb{P}^n = (k^{n+1} \setminus \{\vec{0}\})/k^x = \{\vec{a} := [a_0 : \cdots : a_n] | a_i \in k, \forall i, \vec{a} = \lambda \vec{a}, \forall \lambda \in k^x\}.$$

- Projective Algebraic set

$$V(f_1, \cdots, f_r) = \{\vec{a} \in \mathbb{P}^n : f_1(\vec{a}) = \cdots = f_r(\vec{a}) = 0\}$$

- Zariski topology on $\mathbb{P}^n$: Taking the projective algebraic sets in $\mathbb{P}^n$ as closed sets.
- Irreducible: For any topological space X, $S \subseteq X$ is ***reducible*** if $\exists$ nonempty proper closed sets $C_1, C_2 \subseteq X$ with $S = C_1 \cup C_2$. Otherwise we call S is ***irreducible***.
- Projective Variety: A closed irreducible subsets of $\mathbb{P}^n$.
- Quasiprojective Variety (QP-variety): $X \subseteq \mathbb{P}^n$ is of the form

$$X = Z \cap U, U \subseteq \mathbb{P}^n \text{ open, } Z \subseteq \mathbb{P}^n \text{ a projective vareity .}$$

  Note that QP-varieties are irreducible (and dense); see [Har97][Example 1.13].

Example 1.1 (Important QP-Varieties)**.** *1. Projective varieties* ($U = \mathbb{P}^n$).

2. Affine charts. Let $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\} = V(x_i)^c \cong \mathbb{A}^n$. *This is complement of a variety, so open. Now let* $U = U_i, Z = \mathbb{P}^n$. *Then,*

$$\mathbb{P}^n = U_0 \cup \cdots \cup U_n.$$

3. Affine Varieties: $U = U_i, Z$ *be any projective variety. Note that coordinate ring is* $K[x_0, \cdots, x_n]/I$ *for some ideal, thus it is finitely generated* k*-algebras that are integral domain.*

Note that they are all coordinate dependent.

Definition 1.2 (Morphism)**.** $\varphi : X \subseteq \mathbb{P}^n \to Y \subseteq \mathbb{P}^m$ *is* **morphism** *of QP-variety, if it is*

1. *Zariski continuous*
2. $\forall V \in Top(Y), \forall f \in \mathcal{O}_Y(V)$,

$$f \circ \varphi \in \mathcal{O}_X(f^{-1}(V)) \text{ where } f \circ \varphi : \varphi^{-1}(V) \to k$$

Note that second condition means that pull back of regular function is also regular. To define notation used for morphism precisely, we need bunch of definitions...

1. Function field:

$$\mathcal{K}(X) = \{[f, U] : U \in Top(X), f : U \to K, f = \frac{g}{h}, g, h \in k[x_0, \cdots, x_n] \text{ are homogeneous of the same degree.}\}$$

   where $[f, U]$ is equivalent class of functions so that

$$[f, U] \cong [g, V] \text{ if } \exists W \in Top(X) \text{ s.t. } W \subseteq U \cap V \text{ and } f|_W = g|_W.$$

   Thus,

$$f(\lambda \vec{x}) = \frac{g(\lambda \vec{x})}{h(\lambda \vec{x})} = \frac{\lambda^d g(\vec{x})}{\lambda^d h(\vec{x})} = \frac{g(\vec{x})}{h(\vec{x})} = f(\vec{x}).$$

2. Local ring at $p \in X$:

$$\mathcal{O}_{X,p} := \{[f, U] \in \mathcal{K}(U) : p \in U\}$$

3. Ideal of functions vanishing at p:

$$m_p := \{[f, U] \in \mathcal{O}_{X,p} : f(p) = 0\}.$$

4. Ring of regular functions on $V \in Top(X)$

$$\mathcal{O}_X(V) = \{[f, U] \in \mathcal{K}(x) : V \subseteq U\} = \bigcap_{p \in V} \mathcal{O}_{X,p}.$$

5. Global functions(sections):

$$\mathcal{O}_X(X) = \{[f, U] \in \mathcal{K}(x) : X \subseteq U\} = \{[f, X] \in \mathcal{K}(x)\} \subseteq \mathcal{K}(X).$$

   See section 1.3 theorem 3.2 (affine) and 3.4 (projective) to see that global section of affine variety is just coordinate ring, and that of projective variety is just k.

Now why do we care all about this? since ***All of this information is encoded on Sheaf of** X**.*** Note that every QP-variety X is the union of affine varieties, since

$$X = Z \cap U = (Z \cap U_0) \cup \cdots \cup (Z \cap U_n) \cap U$$

so it "is locally an affine variety."

Affine Scheme is local object, so we can say scheme is "locally affine." One of the advantage of scheme is that it does not gives any restriction; we can think of any ring as geometric object, compared with restrictions of coordinate ring, a finitely generated k-algebra with integral domain. Thus it allows us to distinguish $V(x) = V(x^2) = V(x^3)$. For example, every complement of hypersurface is embedded as variety is one of Hartshorne's example, so from this we can embed $\mathbb{R} \setminus \{0\}$ to $V(xy - 1)$. You can see this in [Har97][Prop 4.9 in Section 1.4].

1.2 Chapter 2.1

Definition 1.3 (Presheaf, ver 1)**.** *A* **presheaf** *(of Abelian group) on topological space X is*

1. *(Object) $\forall U \in Top(X)$, $\mathcal{F}$ assign an abelian group $\mathcal{F}(U)$.*
2. *(Arrow) $\forall U, V \in Top(X)$ with $U \subseteq V$, $\mathcal{F}$ assigns a group homomorphism*

$$\mathrm{res}_{vu} : \mathcal{F}(V) \to \mathcal{F}(U) \text{ satisfying}$$

   (a) *(Preserve identity)* $\mathrm{res}_{uu} = id_{\mathcal{F}(U)}, \forall U \in Top(X)$,
   (b) *(Preserve composition) If $U \subseteq V \subseteq W$, then* $\mathrm{res}_{WU} = \mathrm{res}_{VU} \circ \mathrm{res}_{WV}$.

3. $\mathcal{F}(\emptyset) = \{0\}$.

Definition 1.4 (Presheaf, ver 2)**.** *A* **presheaf** *is a contravariant functor* $\mathcal{F} : \mathbf{Top(X)} \to \mathbf{Ab}$.

Definition 1.5 (Sheaf)**.** *A* **sheaf** *is a presheaf $\mathcal{F}$ satisfying*

1. *(Identity) If $\{U_i\}$ is open cover for $U \in Top(X)$, $f \in \mathcal{F}(U)$ satisfying $\mathrm{res}_{UU_i}(f) := f|_{U_i} = 0$ for all i, then $f = 0$. Equivalently, if $f, g \in \mathcal{F}(U)$ with $f|_{U_i} = g|_{U_i}$ for all i, then $f = g$.*
2. *(Gluing) If $\{U_i\}$ is an open cover of $U \in Top(X)$ and for all i, $f_i \in F(U_i)$ satisfies*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}, \forall i, j$$

   then $\exists f \in F(U)$ such that $f|_{U_i} = f_i$.

Note that $F(U)$ is called **section** *of U, and $F(X)$ is called* **global sections***.*

We use notation of restriction to denote

$$res_{UU_i} = f|_{U_i}.$$

Actually, if the abelian group consists of function, then *res* is just usual restriction of function.

Example 1.6 (Examples and nonexamples of Sheaf)**.**

1. *(Structure sheaf on a variety) X = QP-variety. Then $U \mapsto \mathcal{O}_X(U)$ is sheaf and* res *is just usual restriction of function.*
2. *X be any topolgical space. Then $U \mapsto C^0(U, \mathbb{R})$ is sheaf.*
3. *(Structure sheaf on a manifold) X is a smooth manifold, then $U \mapsto C^\infty(U, \mathbb{R})$ is sheaf.*
4. *(Counter-ex) Let $X = \mathbb{R}$, and $BC(U, \mathbb{R})$ be a set of bounded functions from U to $\mathbb{R}$. Then $U \mapsto BC(U, \mathbb{R})$ is not a sheaf, since even if $BC([n, n+1], \mathbb{R})$ is nontrivial, so it contains identity function, but $BC(\mathbb{R}, \mathbb{R})$ doesn't contain identity, so it cannot satisfy gluing condition.*
5. *(Counter-ex) Let $X = \{0, 1\}$ with discrete topology. Let*

$$\mathcal{F}(\{0,1\}) = \mathbb{R}^3, \mathcal{F}(\{0\}) = \mathbb{R}, \mathcal{F}(\{1\}) = \mathbb{R}, \mathcal{F}(\emptyset) = \{0\}$$

and

$$\mathrm{res}_{X,\{0\}} = \pi_1 : (a, b, c) \mapsto a, \mathrm{res}_{X,\{1\}} = \pi_b : (a, b, c) \mapsto b,$$

and all other res *to be zero map. Then, it is not a sheaf since it cannot satisfying gluing condition. (See $2 \in \mathcal{F}(\{0\}), \pi \in \mathcal{F}(\{1\})$.)*

2 Sep 19, 2018: Chapter 2.2

Lectured by C. J. Bott,

2.1 Recall from last

(Recall from last time)
Motivation: To study SCHEMES, an important and FUN generalization of quasiprojective varieties.

1. Quasiprojective varieties are "locally affine" (unions of affine varieties)
2. Quasiprojective varieties have a **sheaf** as part of their definition:

$$\mathcal{O}_X(X): \text{ global sections}, \mathcal{O}_X(u): \text{ regular functions}, \mathcal{O}_{X,p}: \text{ local ring}$$

3. Schemes will be "locally affine" (unions of affine schemes) and also have a sheaf.
4. Quasiprojective varieties required an embedding in $\mathbb{P}^n$, schemes are coordinate-free (intrinsic)
5.

$$\{\text{Affine Varieties over } k\} \cong \{\text{Fin. gen. } k\text{-algebra that are integral domains }\}$$

and

$$\{\text{affine schemes}\} \cong \{\text{Rings}\}$$

Recall definition of presheaf and sheaf.

2.2 Rings of functions (structured sheafs) and Nonexamples

Note that res. is usual restriction in these sheafs. Also, X is space, $U \in Top(X)$.

1. $X =$ Quasi projective varieties, $\mathcal{O}_X(U) =$ regular function of U
2. X is any topological space, sheaf is $U \mapsto C^0(X;\mathbb{R})$, space of continuous function from X to $\mathbb{R}$.
3. X is smooth manifold, sheaf is $U \mapsto C^\infty(X;\mathbb{R})$, space of smooth function from X to $\mathbb{R}$.

Also there are several nonexample.

1. $X = \mathbb{R}, u \mapsto BC(U;\mathbb{R})$ is nonexample, since boundedness is not local property. Consider identity function on each $(n, n+2)$ for all $z \in \mathbb{Z}$.
2. $X = \{0,1\}$ with discrete topology, and presheaf $\mathcal{F}$ is $\mathcal{F}(X) = \mathbb{R}^3$ with projection onto first element, $\mathcal{F}(0) = \mathcal{F}(1) = \mathbb{R}$, projection onto second element. (See counterexample 5 in previous section.)
3. (Constant Presheaf) Fix A be abelian group. Let $F(U) = A$. Define

   $$\text{res}(U) := \begin{cases} id & \text{o.w.} \\ 0 & \text{if } U = \emptyset. \end{cases}$$

   Then it is not sheaf in general; to see this, let $X = \{0,1\}$ with discrete topology, $A = \mathbb{Z}$. Then, $F(\{1\}) = \mathbb{Z} = F(\{0\})$. Hence $2 \in F(\{0\}), 3 \in F(\{1\})$. This implies

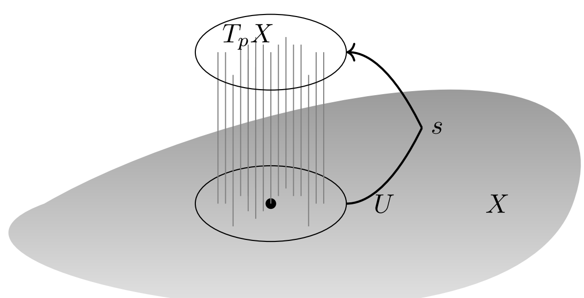
   $$2|_{\{0\}\cap\{1\}} = 3|_{\{0\cap 1\}} = 0.$$

   Hence from the gluing axiom, $\exists n \in \mathbb{Z}$ such that $n|_{F(\{0\}} = 2, n|_{F(\{1\})} = 3$, but there is no such integer.

2.3 Sections of a Vector Bundle

1. X is smooth manifold, and $U \mapsto \Gamma^\infty(U)$, where $\Gamma(U)$ is the smooth vector field on U, i.e.,

   $$s \in \Gamma^\infty(U) \implies s : U \to \sqcup_{p\in U} T_pX \text{ s.t. } s(p) \in T_p(X).$$

   As you can see the picture, sheaf can be regarded as a set of such function s, which is "sheaf" in "natural (or agricultural)" sense, if you can find a kind of similarity with wheat's sheaf. Now the "stalk" which we will give definition today, is analyzing main stems near T_pX, which is "stalk in sheaf" in agricultural sense. So, it is quite natural :)

   Also note that $\Gamma^\infty(U)$ is a $C^\infty(U;\mathbb{R})$-module by multiplication.

2. X is smooth manifold, and $U \mapsto \Omega^k(U)$, space of differential k form on U, i.e.,

   $$s \in \Omega^k(U) \implies s : U \to \sqcup_{p\in U} \wedge^k (T_pX^*)$$

   where $s(p) \in \wedge^*(T_pX^*)$, s is smooth. Note that T_pX^* is cotangent space of X^*.

2.4 Other implication

1. Constant Sheaf (associated to constant presheaf) Let $A \in \mathbf{Ab}$, the category of abelian group. Then sheaf is
$$U \mapsto C^0(U; A),$$
where A is given discrete topology.

2. Skyscraper sheaf: let $A \in \mathbf{Ab}, p \in X$ is a point fixed. Then, sheaf $\mathcal{F}$ is
$$\mathcal{F}\iota_{p,*}A(U) := \begin{cases} A & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}.$$

2.5 Stalks

Definition 2.1 (Stalks)**.** *For $p \in X$, $\mathcal{F}$ a presheaf, the* **stalk** *of $\mathcal{F}$ at p is*
$$\mathcal{F}_p := \varinjlim_{\substack{U \in Top(X) \\ p \in U}} \mathcal{F}(U),$$
i.e. direct limit (special case of colimit in category theory). Concretely,
$$\mathcal{F}_p = \varinjlim_{\substack{U \in Top(X) \\ p \in U}} \mathcal{F}(U) := \{[f, U] : p \in U \in Top(X), f \in \mathcal{F}(U)\}/\sim$$
where
$$[f, U] \sim [g, V] \text{ if } \exists p \in W \subseteq U \cap V \text{ such that } f|_W = g|_W.$$

Claim 2.2.

1. *$[f, U] = [f|_W, W]$ for all $W \subseteq U$.*
2. *$\mathcal{F}_p$ is an abelian group, $\forall p \in X$.*

Proof. 1) is just trivially application of the equivalence relation. To see 2), note that
$$[f, U] + [g, V] = [f|_{U\cap V} + g|_{U \cap V}, U \cap V] \text{ (Additivity)}$$
and note that
$$[0, U] \text{ for any } p \in U.$$
For the last statement, note that $[f, U] = 0 \implies \exists W \subseteq U$ such that $p \in W, f|_W = 0$. □

Remark 2.3. *$[f, U] \in \mathcal{F}_p$ is called a* **germ**, *"big seeds" in "stalk."*

Example 2.4 (Example of stalks in previous cases)**.**

1. *For sheaf from smooth manifold to a smooth vector field, stalk is just T_pX.*
2. *For sheaf from smooth manifold to a space of differential k-forms, stalk is just T_pX^*.*
3. *For a constant sheaf in previous example, stalk is A.*
4. *For a skyscraper sheaf, stalk is either A or 0; it is A when $Q \subseteq \overline{\{p\}}$, a closure of point in Zariski topology, and zero otherwise.*

Let $\mathcal{F}, \mathcal{G}$ are presheaves.

Definition 2.5 (Morphism of presheaves)**.** *A* **morphism of presheaves** $\varphi : \mathcal{F} \to \mathcal{G}$ *is a collection of group homomorphism*

$$\{\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)\}$$

such that if $U \subseteq V$, then below diagram commutes.

$$\begin{tikzcd}
f \in \mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "res:\ |_U"] & \varphi_V(f) \in \mathcal{G}(V) \arrow[d, "res:\ |_U"] \\
f|_U \in \mathcal{F}(U) \arrow[r, "\varphi_U"] & \varphi_U(f|_U) = \varphi_V(f)|_U \in \mathcal{G}(U)
\end{tikzcd}$$

i.e.,

$$\varphi_U(f|_U) = \varphi_V(f)|_U, \forall f \in \mathcal{F}(V).$$

Definition 2.6 (Morphism of presheaves, in categorical sense)**.** *φ is a natural transformation of two sheafs.*

Thus, φ is not depends on U, V.

Definition 2.7 (Morphism of sheaves)**.** *A* **morphism of sheaves** *is a morphism of presheaves.*

Remark 2.8 (Isomorphism)**.** *A morphism $\phi : \mathcal{F} \to \mathcal{G}$ is isomorphism if ϕ_U is isomorphism for all $U \in Top(X)$.*

Proposition 2.9. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves there is an induced group homomorphism on stalk $\varphi_P : \mathcal{F}_p \to \mathcal{G}_p$, $\forall p \in X$.*

Proof. Define $\varphi_p([f, U]) := [\varphi_U(f), U]$ for each $[f, U] \in F(U)$, $[\varphi_U(f), U] \in G(U)$. It suffices to show that it is well-defined.

If $[f, U] = [g, V]$, then

$$\begin{aligned}
& \exists p \in W \subseteq U \cap V \text{ s.t. } f|_W = g|_W \\
\Longrightarrow\ & \varphi_W(f|_W) = \varphi_W(g|_W) && \text{from homomorphism } \varphi_W \\
\Longrightarrow\ & \varphi_U(f)|_W = \varphi_V(g)|_W && \text{by commutative diagram} \\
\Longrightarrow\ & [\varphi_U(f), U] = [\varphi_V(g), V] && \text{by equivalence relation in the stalk} \\
\Longrightarrow\ & \varphi_p([f, U]) = \varphi_p([g, V]) && \text{by construction of } \varphi_p.
\end{aligned}$$

□

Theorem 2.10. *$\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism of sheaves $\iff$ The induced map $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ are isomorphisms $\forall p \in X$. i.e.,*

$$\phi_U : \mathcal{F}(U) \xrightarrow{\cong} \mathcal{G}(U), \forall U \in Top(X) \iff \varphi_p : \mathcal{F}_p \xrightarrow{\cong} \mathcal{G}_p, \forall p \in X.$$

Proof. ($\Rightarrow$) Assume $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is isomorphism fo all $U \in Top(X)$. Then

1. φ_p is surjective. Let $[g, V] \in \mathcal{G}_p$. Then,

   $$g \in \mathcal{G}(V) \implies \exists f \in \mathcal{F}(V) \text{ s.t. } \varphi_V(f) = g$$

   since φ_V is isomorphism. Hence,

   $$\varphi_p([f, V]) = [\varphi_V(f), V] = [g, V].$$

2. φ_p is injective. Assume $\varphi_p([f, U]) = [\varphi_U(f), U] = \overline{0} \in G_p$. Then,

   $$\exists W \subseteq U \text{ s.t. } p \in W, \varphi_U(f)|_W = 0.$$

   Then by commutativity,

   $$\varphi_U(f)|_W = 0 \implies \varphi_W(f|_W) = 0 \implies f|_W \text{ since injectivity} \implies [f, U] = 0 \in \mathcal{F}_p.$$

Note that ($\Rightarrow$) holds for presheaves.

($\Leftarrow$). Assume φ_p is isomorphism, $\forall p \in X$.

1. φ_U is injective. Assume that $\varphi_U(f) = 0$. This implies

$$[0, U] = [\phi_U(f), U] = \overline{0} \in \mathcal{G}_p, \forall p \in U \implies \varphi_p([f, U]) = [\varphi_U(f), U] = \overline{0}.$$

   Since φ_p is injective, $[f, U] = \overline{0} \in \mathcal{F}_p$ for all $p \in X$. Hence,

$$\forall p \in X, \exists W_p \subseteq U \text{ such that } p \in W_p, f|_{W_p} = 0.$$

   Since $\{W_p\}$ is an open cover for U, by the identity axiom of sheaf, $f = 0$, as desired.

2. φ_p is surjective. Let $g \in \mathcal{G}(U)$. We want to show that $\exists f \in \mathcal{F}(U)$ such that $\varphi_U(f) = g$. Note that $[g, U] \in \mathcal{G}_p, \forall p \in U$. From the isomorphism,

$$\forall p \in U, \exists (f_p, V_p) \in \mathcal{F}(V_p) \times Top(X), \text{ such that } \varphi_p([f_p, V_p]) = [\varphi_{V_p}(f_p), V_p] = [g, U].$$

   This implies that

$$\forall p \in U, \exists W_p \subseteq V_p \cap U \text{ such that } p \in W_p, \varphi_p(f_p)|_{W_p} = g|_{W_p}.$$

   Hence,

$$\varphi_{W_p}(f_p|_{W_p}) = g|_{W_p}, \forall p \in U.$$

   Now for any $p, q \in U$,

$$\varphi_{W_p}(f_p|_{W_p \cap W_q}) = (g|_{W_p})|_{W_p \cap W_q} \implies \varphi_{W_p \cap W_q}(f_p|_{W_p \cap W_q}) = g|_{W_p \cap W_q}$$

   by the commutativity of natural transformation φ_{W_p}. Similarly, for W_q, we can get

$$\varphi_{W_p \cap W_q}(f_q|_{W_p \cap W_q}) = g|_{W_p \cap W_q}$$

   By injectivity of $\varphi_{W_p \cap W_q}$, which we proved above,

$$f_p|_{W_p \cap W_q} = f_q|_{W_p \cap W_q}.$$

   Hence, from the gluing axiom of sheaf $\mathcal{F}$,

$$\exists f \in \mathcal{F}(U) \text{ s.t.} f|_{W_p} = f_p|_{W_p}$$

   In other word,

$$\varphi_U(f|_{W_p} \underbrace{=}_{\text{commutative diagram}} \varphi_{W_p}(f|_{W_p}) \underbrace{=}_{\text{above equation}} \varphi_{W_p} = g|_{W_p}, \forall p \in U.$$

   Thus, by the identity axiom of the sheaf G,

$$\phi_u(f) = g,$$

   done.

$\square$

3 After SWAG: Cateogories and Homological Algebra

Lectured by PABLO, S.

Definition 3.1. *Let C be a category, define C^{op} as the category*

- $ob(C^{op}) = ob(C)$
- $\mathrm{Hom}_{C^{op}}(X, Y) := \mathrm{Hom}_C(Y, X)$

This is called the opposite category.

Notation: $\phi : X \to Y$ such that $\mathrm{Ker}(\phi) = \{x : \phi(x) = 0\}, \ker(\phi) : \mathrm{Ker}(\phi) \to X$.

Definition 3.2. *Given C as category and $\{X_i\}_{i\in I} \subseteq ob(C)$ Consider the functor $C^{op} \to \mathbf{Set}$ by $Y \mapsto \prod_i Hom_{C^{op}}(Y, X_i)$ and $\mathbb{C} \to \mathbf{Set}$ by $Y \mapsto \prod_i \mathrm{Hom}_C(X_i, Y)$.*

If we define it for $G : C \to \mathbf{Set}$, then $G(Y) \cong \mathrm{Hom}_C(X, Y)$ functorially.

Definition 3.3. *We say that functor $\mathcal{F} : C^{op} \to \mathbf{Set}$ is representable if there exists $X \in C$ with*

$$\mathcal{F}(Y) \cong \mathrm{Hom}_C(Y, X).$$

Functorially, i.e., given $\phi \in \mathrm{Hom}_{C^{op}}(Y, Z)$ then

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\mathcal{F}(\phi)"] \arrow[d, "\cong"] & \mathcal{F}(Z) \arrow[d, "\cong"] \\
g \in \mathrm{Hom}_C(Y, X) \arrow[r] & g \circ \phi \in \mathrm{Hom}_C(Z, X)
\end{tikzcd}$$

So it is naturally isomorphic to $\mathrm{Hom}(-, X)$.

Lemma 3.4 (Yoneda Lemma)**.** $\mathrm{Hom}_{C^\wedge}(h_C(X), A) \cong A(X)$*, functorial, where $C^\wedge :=$ Functors $(C^{op}, \mathbf{Set})$, and for $A \in C^\wedge$, $X \in C, h_C : C \to C^\wedge$ by $X \to \mathrm{Hom}_C(-, X)$.*

Definition 3.5. *Whenever product and or coproduct are representable, then we call $\prod_i X_i$ and $\coprod_i X_i$ their representatier.*

For any choice of Y and any $\phi_j : Y \to X_j$, Given $\{X_i\}_{i\in I}$, we want to build $\prod_{i\in I} X_i$ For this i, we need

$$\begin{tikzcd}
 & & X_j \\
Y \arrow[r, dashed] \arrow[urr, bend left, "\phi_j"] \arrow[drr, bend right, "\phi_k"'] & \prod_{i\in I} X_i \arrow[ur, "\prod_j"'] \arrow[dr, "\prod_k"] & \\
 & & X_k
\end{tikzcd}$$

Also for coproduct,

$$\begin{tikzcd}
 & & X_j \arrow[dl, "\iota_j"] \arrow[dll, bend right, "\phi_j"'] \\
Y & \coprod_{i\in I} X_i \arrow[l, dashed] & \\
 & & X_k \arrow[ul, "\iota_k"'] \arrow[ull, bend left, "\phi_k"]
\end{tikzcd}$$

Definition 3.6. *For any choice of Y and any ϕ_j, Consider I a category, $\beta : I^{op} \to C$ a functor. Then* **projective limit** $\varprojlim \beta(i)$ *(or* **direct limit** *in the sense we can deal with directed set, or just* **colimit** *if we deal with (or think directed set as) small category) is an object satisfying below commuting diagram;*

$$\begin{tikzcd}
 & & \beta(j) \\
Y \arrow[urr, bend left, "\phi_j"] \arrow[drr, bend right, "\phi_k"'] \arrow[r, dashed] & \varprojlim \beta \arrow[ur, "\Pi_j"'] \arrow[dr, "\Pi_k"] & \\
 & & \beta(k) \arrow[uu, "\beta(s)"']
\end{tikzcd}$$

for any $s : j \to k$, $j, k \in ob(I)$. Note that in the category of **Set**, $\varprojlim \beta$ *is just* $\prod_{i \in I} X_i$.

Similarly, if $\alpha : I \to C$ is given, then **injective limit** $\varinjlim \beta(i)$ *(or* **inverse limit** *in the sense we can deal with directed set, or just* **limit** *if we deal with (or think directed set as) small category) is an object satisfying below commuting diagram;*

$$\begin{tikzcd}
 & & \beta(j) \arrow[dll, bend right, "\phi_j"'] \arrow[dl, "\iota_j"] \\
Y & \varinjlim \beta \arrow[l, dashed] & \\
 & & \beta(k) \arrow[ull, bend left, "\phi_k"] \arrow[ul, "\iota_k"'] \arrow[uu, "\beta(s)"']
\end{tikzcd}$$

Definition 3.7 (Cone, cocone)**.** *A* **cone** *of the functor $F : B \to C$ is an object N together with morphisms $\{\phi_b : N \to F(b)\}_{b \in B}$ such that for any $b_i, b_j \in B$, and for every morphism $b_i \to b_j$, below diagram commutes*

$$\begin{tikzcd}
N \arrow[r, "\phi_{b_i}"] \arrow[dr, "\phi_{b_j}"'] & b_i \arrow[d] \\
 & b_j
\end{tikzcd}$$

Definitely, we can define cocone.

Then, projective limit is just a universal cone, and injective limit is just a universal cocone.

Example 3.8.

1. *Let I be the category with two objects and two parallel morphisms other than identities, visualized by*

$$\bullet \rightrightarrows \bullet$$

A functor $\alpha : I \to C$ is just give a map f, g in $\mathrm{Hom}_C(X_0, X_1)$, for some objects $X_0, X_1 \in C$ with.

$$f, g : X_0 \rightrightarrows X_1.$$

By definition of kernel (or equalizer) in category theory, we get

$$Ker(f,g) \xrightarrow{k} X_0 \overset{g}{\underset{f}{\rightrightarrows}} X_1$$

such that $g \circ k = f \circ k$ and for any morphism $m : Y \to X_0$ with $f \circ m = g \circ m$, then it has unique morphism $eq : Y \to Ker(f,g)$ s.t. $eq \circ k = m$. Thus, *By drawing commutative diagram for this universal property for any cone Y over α, we can check that it is just projective limit.*

$$\begin{tikzcd}
 & & X_1 \\
Y \arrow[r, dashed] \arrow[urr, bend left, "\phi_1"] \arrow[drr, bend right, "m"'] & \ker(f,g) \arrow[ur, "h"] \arrow[dr, "k"'] & \\
 & & X_0 \arrow[u, shift left, "g"] \arrow[u, shift right, "f"']
\end{tikzcd}$$

Note that $\phi_1 = f \circ m = g \circ m$ by definition of cone, and $h := f \circ k = g \circ k$.

2. *When I has only one object and identity is the only arrow, then limit (left arrow) and colimit (Right arrow) coincides with product and coproduct.*

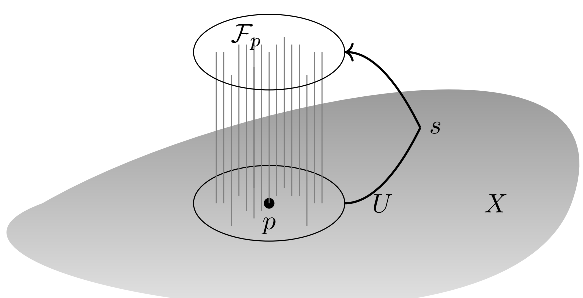
   Proof. To see this, suppose that $I = \{X_i\}_{i \in \mathcal{I}}$. note that cone of α is any object N having morphisms $\{N \to \alpha(X_i)\}_{i \in \mathcal{I}}$. Hence, product just satisfy commutative diagram of limit vacuously. $\square$

3. *Let I be the empty category. THen lim(left arrow) is a terminal object and right arrow one is initial object.*

   Proof. In this case, every element in C is a cone over α vacuously; hence if there is universal cone, which satisfy only one property that there exists morphism from all cones in C to that universal object, done. And we call such object as the terminal object. $\square$

4 Sep 26, 2018: Sheafification

Lectured by C. J. Bott, Recall below picture;



Note that section in Sheaf is a map from $U \to \sqcup_{p \in U} \mathcal{F}_p$ such that

1. $s(p) \in \mathcal{F}_p$
2. s to be continuous, differentiable, etc.. depending on the context of object X
3. If we think $\pi_p : \mathcal{F}_p \to \{p\}$, then
$$\sqcup_{p \in U} \pi_p \circ s = id.$$

In general, section in presheaf is just an element of $\mathcal{F}(U)$, and we can give a map on each section to make presheaf to sheaf. (Sheafification!) For example, if we have constant presheaf on a object A with discrete topology, such as,

$$A_{pre}(U) = \begin{cases} A & \text{if } U \neq \emptyset \\ 0 & \text{o.w.} \end{cases}$$

, then its sheafification $\underline{A}(U) = C^0(U; A)$. So $f^{-1}(a)$ should be clopen set for any $a \in A$, i.e., f is locally constant.

Remark 4.1 (Name of several sheaf-like objects)**.** **Separated presheaf** *is a presheaf satisfying identity axiom of the sheaf. And it is know that an object satisfying condition 1,2,4,5 of the sheaf is a sheaf, i.e., condition 1,2,4,5 implies 3. And condition 4 is needed to show uniqueness in condition 5.*

Definition 4.2 (Kernel, Image, Cokernel)**.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then,*

1. $\ker(\varphi) : U \mapsto \ker(\phi_U) \subseteq \mathcal{F}(U)$.
2. $\operatorname{Im}(\varphi) : U \mapsto \operatorname{Im}(\phi_U) \subseteq \mathcal{G}(U)$.
3. $\operatorname{coker}(\varphi) : U \mapsto \operatorname{coker}(\phi_U) \subseteq \mathcal{G}(U)/\operatorname{Im}(\phi_u)$.

Proposition 4.3. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves,*

1. $\ker(\varphi)$ *is a sheaf*
2. $\operatorname{Im}(\varphi)$ *is a sheaf if ϕ_U is injective for all U,*
3. $\operatorname{coker}(\varphi)$ *is a sheaf if ϕ_U is injective for all U.*

Proof. 1) First of all, $\ker(\varphi_U)$ is an abelian group.

2) Also, if $W \subseteq U$, then $res_{UW} : \mathcal{F} \to \mathcal{F}(W)$ gives

$$res_{UW}|_{\ker(\varphi_U)} : \ker(\varphi_U) \to \mathcal{F}(W).$$

Now we need to show that $\operatorname{Im} res_{UW}|_{\ker(\varphi_U)} \subseteq \ker(\phi_W)$. To see this, note that

$$f \in \ker(\varphi_U) \implies \varphi_U(f) = 0 \implies \varphi_U(f)|_W = 0|_W = 0 \implies \varphi_W(f|_W) = 0.$$

And from the commutativity of morphism, this implies $\varphi_U(f)|_W = 0$, hence $\operatorname{Im} res_{UW}|_{\ker(\varphi_U)} \subseteq \ker(\varphi_W)$, as desired.

3) $\ker(\varphi_\emptyset) = 0$ since $\emptyset$ is just $0 \to 0$.

4) If $\{U_i\}$ is an open cover for U and $f \in \ker(\varphi_U)$ satisfies $f|_{U_i} = 0$ for all i, then $f = 0$ since $\mathcal{F}$ is a sheaf.

5) Let $\{U_i\}$ be an open cover for U and let $f_i \in \ker(\varphi_{U_i} \subseteq \mathcal{F}(U_i)$ satisfy $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ fo any i, j. Since $\mathcal{F}$ is a sheaf, $\exists f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all i, hence it suffices to show that $f \in \ker(\varphi_U)$. Note that

$$\varphi_U(f)|_{U_i} = \varphi_{U_i}(f|_{U_i}) = \varphi_{U_i}(f_i) = 0, \forall i.$$

where first equality comes from the commutativity of diagram of the morphism of presheaves, and the second equality comes from restriction, and third equality comes from the construction of f_i, done. Hence by identity axiom, $\varphi_U(f) = 0$. □

Definition 4.4. *Let $\mathcal{F}$ be a presheaf. The* **sheafification** $(\mathcal{F}^{sh}, \theta : \mathcal{F} \to \mathcal{F}^{sh})$ *of $\mathcal{F}$ is the sheaf (if it exists) that satisfies the following universal property; if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism from $\mathcal{F}$ to a sheaf $\mathcal{G}$, then there exists unique morphism of sheaf $\Phi : \mathcal{F}^{sh} \to \mathcal{G}$ such that below diagram commutes.*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & \mathcal{F}^{sh} \arrow[d, "\exists!\Phi"] \\ & \mathcal{G} \end{tikzcd}$$

And actually, we can sheafify using this construction!

$$\mathcal{F}^{sh}(U) := \{s : U \to \sqcup_{p \in U} \mathcal{F}_p\}$$

such that

1. $s(p) \in \mathcal{F}_p, \forall p \in U$
2. $\forall p \in U, \exists V$ such that $p \in V \subseteq U, \exists t \in \mathcal{F}(V)$ such that

$$\forall q \in V, s(q) = [t, V]_q$$

and restriction is usual restriction as a function. In other words,

$$\theta_U(f) : U \to \sqcup_{p\in U}\mathcal{F}_p, \theta_U(f)(p) = [f, U]_p \text{ which is a map} f \mapsto [f] \in \mathcal{F}_p.$$

Those two condition is called compatible section.

Remark 4.5.

1. *$\mathcal{F}^{sh}$ is a sheaf.*
2. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \cong \mathcal{F}^{sh}$*
3. *$\mathcal{F}^{sh}$ satisfy the universal property.*
4. *$\mathcal{F}^{sh}_p \cong \mathcal{F}_p$ for all $p \in X$. So $\mathcal{F}^{sh}$ is the "smallest" sheaf with the same stalkes as $\mathcal{F}$.*

Note that if $\mathcal{F}$ satisfies condition 4, then morphism $\mathcal{F} \to \mathcal{F}^{sh}$ is injective.

Definition 4.6. *$\sqcup_{p\in U}\mathcal{F}_p$ with a given topology is called the Espace Étalé. (Exercise 1.13). So we can say that*

$$\begin{aligned}\mathcal{F}^{sh} := \{s : U \to \sqcup_{p\in U}\mathcal{F}_p \ & \textit{with given topology such that} \\ & (1) s(p) \in \mathcal{F}_p, \forall p \in U, \\ & (2) s \textit{ is continuous.}\}\end{aligned}$$

To define the topology, we need sheafification morphism.

Only a presheaf (In general)	Always a sheaf
Image	kernels
Coker	(Finite) direct sums (Ex 1.9)
Quotients	Direct limit when X is Noetherian (Ex 1.11)
Direct limits (Ex 1.10)	Inverse limits (Ex 1.12)
Tensor Product	Sheaf Homomorphism (Ex 1.15)
Inverse Image of (pre)sheaf	$***$ Direct image of sheaf

In case of kernel, for $\phi : \mathcal{F} \to \mathcal{G}$ a sheaf morphism,

$$\ker(\phi)(u) = \ker(\phi(u))$$

so

$$\phi := \{\phi_u : \mathcal{F}(u) \to \mathcal{G}(u)\}.$$

Definition 4.7 (Noetherian space)**.** *A topological space X is* **Noetherian** *if $\forall$ descending chain of closed subsets*

$$X \supseteq \cdots \supseteq U_i \supseteq U_{i-1} \supseteq \cdots$$

stabilizes.

Definition 4.8 (Sheafification morphism)**.** *A sheafification morphism $\theta : \mathcal{F} \to \mathcal{F}^{sh}$ is defined as*

$$\theta_U(f) : U \to \sqcup_{p\in U}\mathcal{F}_p$$

such that

$$\theta_U(f)(p) := [f, U], \forall p \in U \in Top(X), \forall f \in \mathcal{F}(U).$$

So

$$\theta = \{\theta_U : \mathcal{F}(U) \to \mathcal{F}^{sh}(U)\}$$

is compatible with condition 2. (Note that this is just

Construction of a etale space. From sheafification morphism, just define

$$\mathcal{F}^{sh} := \{\theta_U(f) : U \to \sqcup_{p\in U}\mathcal{F}_p\}_{U\in Top(X), f\in\mathcal{F}(U)}$$

(It seems to be tautology, but just forget θ as a presheaf morphism but a set of bunch of sections $U \to \sqcup_{p\in U}\mathcal{F}_p$.)

Now for each $\theta_U(f) \in \mathcal{F}^{sh}$, let

$$\theta_U(f)[U] := \{[f, U]_p \in \mathcal{F}_p \subseteq \sqcup_{p\in U}\mathcal{F}_p | p \in U\}$$

and define $B = \{\theta_U(f)[U]\}_{U\in Top(X), f\in\mathcal{F}(U)}$. Then, define $Top(\sqcup_{p\in U}\mathcal{F}_p)$ be a topoloogy generated by B as a basis.

Thus, let $s = \theta_U(f)$. Then, for any $p \in U$, and for any open subset V of U containing p, there exists $t = f|_V \in \mathcal{F}(U)$, so that

$$\forall q \in V, s(q) = \theta(U)(f)(q) = [f, U]_q = [f|_V, V]_q = [t, V]_q$$

since $s|_V$ and s agrees on $V = U \cap V \subseteq U$. Hence this construction gives condition (2) of $\mathcal{F}^{sh}$.

Also, to see $s = \theta_U(f)$ is continuous, note that for any arbitrary element of basis, say $\theta_V(g)[V]$,

$$s^{-1}(\theta_V(g)[V]) = \begin{cases} 0 & \text{if } f|_W \neq g|_W \text{ for any } W \in Top(X) \\ W & W \subseteq V \cap U \text{ is maximal open set where } f|_W = g|_W \end{cases}$$

Thus, s is continuous. □

Theorem 4.9. *Let $\mathcal{F}$ be a presheaf, and $(\mathcal{F}^{sh}, \theta)$ is a sheafification of $\mathcal{F}$. Then,*

1. *$\mathcal{F}^{sh}$ is a shaef.*
2. *$\mathcal{F}^{sh}_p \cong \mathcal{F}_p$.*
3. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \cong \mathcal{F}^{sh}$.*
4. *$(\mathcal{F}, \theta)$ satisfies the universal property; if G is a sheaf and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism, then $\exists!\Phi : \mathcal{F}^{sh} \to G$ such that $\Phi \circ \theta = \varphi$.*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & \mathcal{F}^{sh} \arrow[d, dashed, "\exists!\Phi"] \\ & G \end{tikzcd}$$

Proof. First of all, $\mathcal{F}^{sh}$ is an abelian group, and by checking all morphisms, we can get it is a separated presheaf. So we should check the gluing axiom. Let $\{U_i\}$ be an open cover for U, and let $\{s_i \in \mathcal{F}(U_i)\}$ satisfying

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}.$$

Define $s \in \mathcal{F}^{sh}(U)$ as

$$s(p) := s_i(p) \text{ if } p \in U_i.$$

Then,

1. $s(p) = s_i(p) \in \mathcal{F}_p$,
2. $\forall p \in U_i \subseteq U$ for some i, $\exists V$ such that $p \in V \subseteq U_i \subseteq U$ and $\exists t = s_i|_V \in \mathcal{F}(V)$ such that

$$\forall q \in V, s(q) = [s|_V, V] = [s_i|_V, V] = [t, V]_q$$

So s is well-defined section on $\mathcal{F}^{sh}(U)$, hence gluing axiom holds.

For (2), let $p \in X$. Define

$$\psi : \mathcal{F}^{sh}_p \to \mathcal{F}_p \text{ by } [s, U] \mapsto s(p).$$

We need to check 1) Well-defined, 2) group homomorphism 3) injectivity and 4)surjectivity.

1. Well-defined: If $[s, U] = [t, V]$ in $\mathcal{F}_p^{sh}$, then $\exists W \subseteq U \cap V$, with $p \in W$ such that $s|_W = t|_W$. In particular, $p \in W$, so $s(p) = t(p)$.
2. Group homomorphism:
$$\psi([s, U] + [t, V]) = \psi([s|_{U\cap V} + t|_{U\cap V}, U \cap V]) = (s|_{U\cap V} + t|_{U\cap V})(p) = s(p) + t(p) = \psi([s, U]) + \psi([t, V]).$$
3. Surjectivity: Let $[t, V] \in \mathcal{F}_p$. Then,
$$t \in \mathcal{F}(V) \implies \theta_V(t) \in \mathcal{F}^{sh}(V).$$
Hence,
$$\psi([\theta_V(t), V]) = \theta_V(t)(p) = [t, V]$$
4. Injectivity: Assume that $\psi([s, U]) = 0 \in \mathcal{F}_p$. Then, $s(p) = 0 \in \mathcal{F}_p$. Thus,
$$\exists V_p \subseteq V \text{ s.t.} p \in V_p, \exists t_p \in \mathcal{F}(V_p)$$
such that
$$\forall q \in V_p, s(q) = [t_p, V_p].$$
In particular,
$$0 = s(p) = [t_p, V_p] \implies \exists W_p \text{ s.t. } p \in W_p \subseteq V_p \subseteq V \text{ and } t_p|_{W_p} = 0,$$
and since it is derived from $s(p) = 0$, we get
$$s|_{V_p} = \theta_{V_p}(t_p).$$
Hence,
$$(s|_{V_p})|_{W_p} = (\theta_{V_p}(t_p))|_{W_p} \implies s|_{W_p} = \theta_{W_p}(t_p|_{W_p}) = \theta_{W_p}(0) = 0.$$
where the second equality of the right equation comes from the universal property of θ as a presheaf morphism. Thus,
$$[s, U] = 0 \in \mathcal{F}_p.$$

Before proving (3), recall the theorem 2.10 in this note. Let $\theta : \mathcal{F} \to \mathcal{F}^{sh}$. Then,
$$\theta_U : \mathcal{F}(U) \to \mathcal{F}^{sh}(U) \text{ by } \theta_U(f)(p) = [f, U]_p.$$
So,
$$\theta_p : \mathcal{F}_p \to \mathcal{F}_p^{sh} \cong \mathcal{F}_p \text{ by } \theta_p([f, U]) = [\theta_U(f), U]$$
Then, from the isomorphism $\psi : \mathcal{F}_p^{sh} \to \mathcal{F}_p$ we showed above,
$$\psi \circ \theta_p([f, U]) = \psi([\theta_U(f), U]) = \theta_U(f)(p) = [f, U]$$
From the fact that $\mathcal{F}_p = \mathcal{F}_p^{sh}$ by construction, θ_p is an identity map for all $p \in U$. Hence, θ is isomorphism by theorem 2.10.

For (4), we will prove this later. $\square$

Definition 4.10 (Injectivitiy, surjectivity, subsheaf, and quotient sheaf)**.** *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheafs. Then,*

- *ϕ is injective if* $\ker(\phi) = 0$*, where* 0 *means a constant sheaf.*
- *ϕ is surjective if* $\mathrm{Im}(\phi)$*, which is defined as sheafification of* $\mathrm{Im}_{pre}(\phi)$ *is isomorphic to G, i.e., there exists $\Phi : G \to \mathrm{Im}(\phi)$ such that $\Phi \circ \iota = \theta$.*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] & \mathrm{Im}_{pre}(\phi) \arrow[r, "\theta"] \arrow[dr, "\iota"'] & \mathrm{Im}(\phi) \\
& & G \arrow[u, dashed, "\exists!\Phi"']
\end{tikzcd}$$

- *$\mathcal{F}' \leq \mathcal{F}$ is a subsheaf if $\mathcal{F}'(U) \leq \mathcal{F}(U)$ as a subgroup for all $U \in Top(X)$, and by consequence of this, $\mathcal{F}'_p \leq \mathcal{F}_p$. It is equivalent to say that an injective morphism $\iota : \mathcal{F}' \to \mathcal{F}$ exists.*
- *Take $\mathcal{F}/\mathcal{F}'_{pre}(U) = \mathcal{F}(U)/\mathcal{F}(U)'$ and do sheafify, then we get the quotient sheaf.*

Example 4.11.

$$\ker(\phi) \leq \mathcal{F}, \operatorname{Im}(\phi) \leq G.$$

Proposition 4.12. *$\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Then,*

1. *ϕ is isomorphism $\iff$ ϕ_U is isomorphism for all $U \in Top(X)$. $\iff$ ϕ_p is isomorphism for all $p \in U$ and all $U \in Top(X)$.*
2. *ϕ is injective $\iff$ ϕ_U is injective for all $U \in Top(X)$ $\iff$ ϕ_p is inejctive. for all $p \in U$ and all $U \in Top(X)$.*
3. *ϕ is surjective $\iff$ ϕ_p is surjective for all $p \in U$, and for all $U \in Top(X)$.*
4. *ϕ_U is surjective for all $U \in Top(X)$ $\implies$ ϕ, ϕ_p are surjective for all $p \in U$, for all $U \in Top(X)$.*
5. *ϕ, ϕ_p are sujective for all p does not imply that ϕ_U is surjective for all $U \in Top(X)$ in general.*

Proof. Notes that $1, 2$ are just derived from the definition of Isomorphism of sheaf and theorem 2.10 in this notes. Surjectivity has a problem, since, actually, we need a injectivity to prove surjectivity; see proof of the theorem 2.10. Actually this statement is just summarize results of proof of theorem 2.10. □

5 Oct 3, 2018

Proposition 5.1. *If a presheaf $\mathcal{F}$ satisfies the identity axiom, (i.e., $\mathcal{F}$ is a separated presheaf), then $\theta : \mathcal{F} \to \mathcal{F}^{sh}$ is injective. (So in this case, $\mathcal{F} \leq \mathcal{F}^{sh}$, and $\mathcal{F}$ is a* **subpresheaf***.)*

Proof. By the definition of sheafification, $\theta_U : \mathcal{F}(U) \to \mathcal{F}^{sh}(U)$ is given by $\theta_U(f)(p) = [f, U]_p$. If $\theta_U(f) = 0$, then $\theta_U(f)(p) = [f, U] = [0, U] = \overline{0} \in \mathcal{F}_p$ for all p. Hence, $\forall p \in U, \exists W_p \in Top(U)$ such that $f|_{W_p} = 0$. Since $\{W_p\}_{p \in U}$ forms an open cover of U, by applying the identity axiom, $f = 0$ on U.

Hence θ_U is injective for any $U \in Top(X)$, thus by above proposition, θ is injective as a presheaf morphism. □

Theorem 5.2. *Let $\mathcal{F}$ be a presheaf, and $(\mathcal{F}^{sh}, \theta)$ is the sheafification of $\mathcal{F}$. Then the below universal property holds; for any sheaf G and morphism $\phi : \mathcal{F} \to \mathcal{G}$, there exists the unique morphism $\Phi : \mathcal{F}^{sh} \to G$ such that $\Phi \circ \theta = \phi$.*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^{sh} \arrow[d, "\exists!\Theta"] \\ & \mathcal{G} \end{tikzcd}$$

Outline of the proof. Let's define

$$\begin{aligned} \Phi_U : \mathcal{F}^{sh}(U) &\to \mathcal{G}^{sh}(U) \cong \mathcal{G}(U) \\ &\text{by } s \mapsto \Phi_U(s) \text{ where } \Phi_U(s) : U \to \sqcup_{p \in U} \mathcal{G}_p \\ &\qquad \text{by } p \mapsto \Phi_U(s)(p) := \phi_p(s(p)). \end{aligned}$$

Also notes that $s \in \mathcal{F}^{sh}(U)$ is a function $U \mapsto \sqcup_{p \in U} \mathcal{F}_p$ by $p \mapsto s(p) \in \mathcal{F}_p, \forall p \in U$.

1. Φ is well-defined morphism.
   (a) Show $\Phi_U(s) \in \mathcal{G}^{sh}(U), \forall U \in Top(X), s \in \mathcal{F}^{sh}(U)$. To see this we need to check that
      i. $\Phi_U(s)(p) \in G_p, \forall p$. To see this, by above definition, $\Phi_U(s)(p) = \phi_p(s(p))$ and $s(p) \in \mathcal{F}_p$, and since $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$, we get $\phi_p(s(p)) \in \mathcal{G}_p$, as desired.

ii. $\forall p \in U, \exists p \in V \subseteq U$ and $\tilde{t} \in V$ such that $\forall q \in V, \Phi_U(s)(q) = [\tilde{t}, V]$. To see this, note that from $s \in \mathcal{F}^{sh}$, by construction $\forall p \in U, \exists p \in V \subseteq U$ and $t \in \mathcal{F}(V)$ such that $\forall q \in V, s(q) = [t, V]$. Now take $\tilde{t} = \phi_V(t)$. Then,

$$\Phi_U(s)(q) = \phi_q(s(q)) = \phi_q([t, V]) =$$

$\square$

6 Oct 31 2018

The algebra-geometry dictionary for affine schemes

Definition 6.1. *Let R be a ring. The* **spectrum of** R

$$Spec(R) = \{[P] : P \trianglelefteq R \text{ is a prime ideal}\}$$

So $\{$Ideals of $R\}$ is related to $\{$ subsets of Spec(R)$\}$ by $V(\cdot), I(\cdot)$.

Definition 6.2.

$$V(J) = \{[p] \in Spec(R) : f([P]) = 0, \forall f \in J\} = \{[P] \in Spec(R) : P \supseteq J\}.$$

by

$$R \to R/P \text{ by } f \mapsto f([P]) = f \text{ mod } P.$$

Definition 6.3. Zariski topology on $Spec(R)$ *Closed subsets are of the form $V(J)$ for $J \trianglelefteq R$*

Definition 6.4.

$$I(S) = \{f \in R : f([P]) = 0, \forall [P] \in S\} = \bigcap_{[P] \in S} P$$

Note that $\forall S \subseteq Spec(R)$, $I(S) \trianglelefteq R$ is a radical ideal.

Theorem 6.5. *$V(\cdot)$ and $I(\cdot)$ satisfy the following properties.*

1. $J_1 \subseteq J_2 \implies V(J_1) \supseteq V(J_2)$ *inclusion reversing*
2. $S_1 \subseteq S_2 \implies I(S_1) \supseteq I(S_2)$ *inclusion reversing*
3. $J_1 \cap J_2 \implies V(J_1) \cup V(J_2)$ *union and intersections*
4. $S_1 \cap S_2 \implies I(S_1) \cap I(S_2)$*union and intersections*
5. $V(I(S)) = \bar{S}$ *how to take closure*
6. $I \trianglelefteq R$ *but* $I \neq R \implies V(I) \neq \emptyset$. *Hilbert Nullstellensatz*
7. $I(V(J)) = Rad(J)$. *Hilbert Nullstelensatz*
8. $V(J) = V(Rad(J))$.
9. $V(J_1) \subseteq V(J_2) \iff Rad(J_1) \supseteq Rad(J_2)$
10. $V(\cdot), I(\cdot)$ *are inverses and give 1-1 correspondence in the following sense*

$$\begin{aligned} Ideal &\iff subset of Spec(R) \\ Radical ideal &\iff closed subsets \\ Prime ideals &\iff irreducible closed subsets \\ Maximal ideals &\iff closed points \end{aligned}$$

Proof of the last statements. Let $S \subseteq Spec(R)$ be irreducible closed subset. We want to show $I(S)$ is prime. Let $fg \in I(S) = \bigcap_{[P]\in S} P$. Thus, $fg \in P$ for all $[P] \in S$. Hence, $f \in P$ or $g \in P, \forall [P] \in S$. Hence,

$$\forall [P] \in S, [P] \in V(< f >) \text{ or } [P] \in V(< g >).$$

So,

$$S = [S \cap V(f)] \cup [S \cap V(g)]$$

So $S \cap V(f)$ is closed, and $S \cap V(g)$ is also closed. Hence, WLOG, $S = S \cap V(f)$. This implies S contained in $V(f)$. Thus,

$$\forall [P] \in S, f \in P \implies f \in \cap_{[P]\in S} P = I(S).$$

Let P be prime, and assume $V(P) = V(J_1) \cup V(J_2)(*)$. Then, $\forall [Q] \in V(P), Q \supset P$ and $(*)$ implies $Q \supset J_1$ or $Q \supset J_2$. In particular, $P \supseteq P$, so P contains J_1 or J_2. WLOG, say $P \supseteq J_1$ Then,

$$V(P) \subseteq V(J_1) \subseteq V(J_1 \cup V(J_2) = V(P)$$

implies $V(P) = V(J_1)$. $\square$

6.1 Affine Schemes Everyone should know (in CJ's opinion)

1. Spec(Field) $= \underbrace{\bullet}_{[(0)]}$
2. Spec(DVR) $= \underbrace{\bullet}_{[(0)]}, \underbrace{\bullet}_{M}$. So $\{M\}$ is maximal, so closed, but $\{(0)\}$ is open.
3. *Spec*$(\mathbb{Z}) = \{[p] : \ p \text{ is prime}\} \cup \{0\}$. But it has no discrete topology; points are not open. Also, (0) is open.
4. *Spec*$(k[x])$, k is algebraically closed. Then, $\{(x-a) : a \in k\} \cup \{(0)\}$. Closure of (0) is whole points, so it has dimension 1, so (0) itself can be regarded 0 dimension intuitively, but not rigorous sense.
5. *Spec*$(\mathbb{R}[x])$ Upper Half space itself and (0). So closure of (0) is 2 dimension.
6. *Spec*$(\mathbb{F}_p)$, *Spec*$(\mathbb{Q}[x])$. SInce both polynomial rings are PID, so $P = (f)$ where f is irreducible, which is related to minimal polynomials can be identified .

   $$Spec(\mathbb{Q}[x]) = \bar{Q}/\sim \cup \{[(0)]\}, Spec(\mathbb{F}_p) = \bar{\mathbb{F}}_p/\sim \cup \{[(0)]\}.$$

   by like this; $\pm i \iff (x^2+1)$, 3rd roots of unity $\iff x^2+x+1$
7. *Spec*$\mathbb{Z}[i]$. $\mathbb{Z} \to \mathbb{Z}[i]$. So, each prime in integer may not be prime in $\mathbb{Z}[i]$. We know that $p \in \mathbb{Z}$ is prime if and only if $p \cong 3 \bmod 4$. And $p = \pi \cdot \pi' \in \mathbb{Z}[i]$ if $p \neq 2$ and $p \cong 1 \bmod 4$, where π, π' are conjugate.

   For example, (3) is (3), (5) is decomposed to $(2+i), (2-i)$... And $(1+i), (2)$ are exceptions of the rules, but prime. And also we have (0).
8. $Spec(\mathbb{Z}[x]) = \begin{cases} [(p)] & p \in \mathbb{Z} \text{ prime} \\ [(f)] & f \in \mathbb{Z}[x] \text{ is irreducible} \\ [(p,f)] & p \in \mathbb{Z} \text{ prime, } f \text{ is irreducible in } \mathbb{F}_p[x] \\ [(0)] & \end{cases}$. first two cases are not maximal. Third one is maximal.

   Arithmetic surfaces, Mumford treasure map, ... Taken picture!

9. $Spec(k[x, y])$, where k algebraic closed. Then

$$Spec(k[x, y]) = \begin{cases} [(0)] & \\ [(x-a, y-b)], a, b \in k & \text{maixmal}\mathbb{A}_k^2\text{-traditional} \\ (f) & f \in k[x, y] \text{ is irreducible} \end{cases}$$

Also take picture!

Two dimensional point $[(0)]$

10. $Spec(\mathbb{C}[x_1, \cdots, x_n])$ is usual points + fat points for every irreducible subvariety.

First think maximal ideals if we think Spec!

References

[Har97] Robin Hartshorne, *Algebraic Geometry, Corrected 8th printing*, Springer-Verlag, 1997.

[BRT75] B. R. Tennison, *Sheaf Theory*, Cambridge University Press, 1975.