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THE ALGEBRA-GEOMETRY DICTIONARY FOR AFFINE SCHEMES

Definition 1.1. *Let R be a ring. The* **spectrum of** R

$$Spec(R) = \{[P] : P \trianglelefteq R \text{ is a prime ideal}\}$$

So {*Ideals of* R} *is related to* { *subsets of Spec(R)*} *by* $V(\cdot), I(\cdot)$.

Definition 1.2.

$$V(J) = \{[p] \in Spec(R) : f([P]) = 0, \forall f \in J\} = \{[P] \in Spec(R) : P \supseteq J\}.$$

by

$$R \to R/P \text{ by } f \mapsto f([P]) = f \text{ mod } P.$$

Definition 1.3. Zariski topology on $Spec(R)$ *Closed subsets are of the form* $V(J)$ *for* $J \trianglelefteq R$

Definition 1.4.

$$I(S) = \{f \in R : f([P]) = 0, \forall [P] \in S\} = \bigcap_{[P]\in S} P$$

Note that $\forall S \subseteq Spec(R)$, $I(S) \trianglelefteq R$ *is a radical ideal.*

Theorem 1.5. $V(\cdot)$ *and* $I(\cdot)$ *satisfy the following properties.*

1. $J_1 \subseteq J_2 \implies V(J_1) \supseteq V(J_2)$ *inclusion reversing*
2. $S_1 \subseteq S_2 \implies I(S_1) \supseteq I(S_2)$ *inclusion reversing*
3. $J_1 \cap J_2 \implies V(J_1) \cup V(J_2)$ *union and intersections*
4. $S_1 \cap S_2 \implies I(S_1) \cap I(S_2)$*union and intersections*
5. $V(I(S)) = \bar{S}$ *how to take closure*
6. $I \trianglelefteq R$ *but* $I \neq R \implies V(I) \neq \emptyset$. *Hilbert Nullstellensatz*
7. $I(V(J)) = Rad(J)$. *Hilbert Nullstelensatz*
8. $V(J) = V(Rad(J))$.
9. $V(J_1) \subseteq V(J_2) \iff Rad(J_1) \supseteq Rad(J_2)$
10. $V(\cdot), I(\cdot)$ *are inverses and give 1-1 correspondence in the following sense*

$$\begin{aligned} Ideal &\iff subset of Spec(R) \\ Radical ideal &\iff closed subsets \\ Prime ideals &\iff irreducible closed subsets \\ Maximal ideals &\iff closed points \end{aligned}$$

Proof of the last statements. Let $S \subseteq Spec(R)$ be irreducible closed subset. We want to show $I(S)$ is prime. Let $fg \in I(S) = \bigcap_{[P]\in S} P$. Thus, $fg \in P$ for all $[P] \in S$. Hence, $f \in P$ or $g \in P, \forall [P] \in S$. Hence,

$$\forall [P] \in S, [P] \in V(<f>) \text{ or } [P] \in V(<g>).$$

So,

$$S = [S \cap V(f)] \cup [S \cap V(g)]$$

So $S \cap V(f)$ is closed, and $S \cap V(g)$ is also closed. Hence, WLOG, $S = S \cap V(f)$. This implies S contained in $V(f)$. Thus,

$$\forall [P] \in S, f \in P \implies f \in \cap_{[P]\in S} P = I(S).$$

Let P be prime, and assume $V(P) = V(J_1) \cup V(J_2)(*)$. Then, $\forall [Q] \in V(P), Q \supset P$ and $(*)$ implies $Q \supset J_1$ or $Q \supset J_2$. In particular, $P \supseteq P$, so P contains J_1 or J_2. WLOG, say $P \supseteq J_1$ Then,

$$V(P) \subseteq V(J_1) \subseteq V(J_1 \cup V(J_2) = V(P)$$

implies $V(P) = V(J_1)$. $\square$

1.1 Affine Schemes Everyone should know (in CJ's opinion)

1. Spec(Field) $=\underbrace{\bullet}_{[(0)]}$
2. Spec(DVR) $=\underbrace{\bullet}_{[(0)]},\underbrace{\bullet}_{M}$. So $\{M\}$ is maximal, so closed, but $\{(0)\}$ is open.
3. $Spec(\mathbb{Z}) = \{[p] : \ p \text{ is prime}\} \cup \{0\}$. But it has no discrete topology; points are not open. Also, (0) is open.
4. $Spec(k[x])$, k is algebraically closed. Then, $\{(x-a) : a \in k\} \cup \{(0)\}$. Closure of (0) is whole points, so it has dimension 1, so (0) itself can be regarded 0 dimension intuitively, but not rigorous sense.
5. $Spec(\mathbb{R}[x])$ Upper Half space itself and (0). So closure of (0) is 2 dimension.
6. $Spec(\mathbb{F}_p)$, $Spec(\mathbb{Q}[x])$. SInce both polynomial rings are PID, so $P = (f)$ where f is irreducible, which is related to minimal polynomials can be identified .

   $$Spec(\mathbb{Q}[x]) = \bar{Q}/\sim \cup\{[(0)]\}, Spec(\mathbb{F}_p) = \bar{\mathbb{F}}_p/\sim \cup\{[(0)]\}.$$

   by like this; $\pm i \iff (x^2+1)$, 3rd roots of unity $\iff x^2+x+1$
7. $Spec\mathbb{Z}[i]$. $\mathbb{Z} \to \mathbb{Z}[i]$. So, each prime in integer may not be prime in $\mathbb{Z}[i]$. We know that $p \in \mathbb{Z}$ is prime if and only if $p \cong 3 \bmod 4$. And $p = \pi \cdot \pi' \in \mathbb{Z}[i]$ if $p \neq 2$ and $p \cong 1 \bmod 4$, where π, π' are conjugate.

   For example, (3) is (3), (5) is decomposed to $(2+i), (2-i)$... And $(1+i), (2)$ are exceptions of the rules, but prime. And also we have (0).
8. $Spec(\mathbb{Z}[x]) = \begin{cases} [(p)] & p \in \mathbb{Z} \text{ prime} \\ [(f)] & f \in \mathbb{Z}[x] \text{ is irreducible} \\ [(p,f)] & p \in \mathbb{Z} \text{ prime, } f \text{ is irreducible in } \mathbb{F}_p[x] \\ [(0)] & \end{cases}$. first two cases are not maximal. Third one is maximal.

   Arithmetic surfaces, Mumford treasure map, ... Taken picture!
9. $Spec(k[x,y])$, where k algebraic closed. Then

   $$Spec(k[x,y]) = \begin{cases} [(0)] & \\ [(x-a, y-b)], a, b \in k & \text{maixmal}\mathbb{A}^2_k\text{-traditional} \\ (f) & f \in k[x,y] \text{ is irreducible} \end{cases}$$

   Also take picture!

   Two dimensional point [(0)]
10. $Spec(\mathbb{C}[x_1, \cdots, x_n])$ is usual points + fat points for every irreducible subvariety.

First think maximal ideals if we think Spec!

References

[Har97] Robin Hartshorne, *Algebraic Geometry, Corrected 8th printing*, Springer-Verlag, 1997.

[BRT75] B. R. Tennison, *Sheaf Theory*, Cambridge University Press, 1975.