Supplement and Solution Manual for Introduction to Commutative algebra

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May 29, 2021

Abstract

This note is based on my practice about Atiyah-MacDonald's book [3]. There may be a lot of errors on it. Use it at your own risk.

Contents

1 Rings and Ideal

Proposition 1.1. *There is a one-to-one order-preserving correspondence between the ideals $\mathfrak{b}$ of A which contains $\mathfrak{a}$, and the ideals $\overline{\mathfrak{b}}$ of $A/\mathfrak{a}$, given by $\mathfrak{b} = \phi^{-1}(\overline{\mathfrak{b}})$.*

Proof. Let

$$\Lambda := \{ \text{ ideal containing } \mathfrak{a}\}, \Delta := \{ \text{ ideal of } A/\mathfrak{a}.\},$$

and $\phi : A \to A/\mathfrak{a}$ canonical projection. Then, first of all we should check that $\phi(\mathfrak{b})$ is ideal when $\mathfrak{b}$ is an ideal.

- If $\overline{x} \in A/\mathfrak{a}, \overline{y} \in \phi(b)$,

$$\overline{xy} = \overline{x} \cdot \overline{y} = \phi(x)\phi(y) = \phi(xy) \in \phi(\mathfrak{b})$$

  since $x \in A, y \in \mathfrak{b}$ implies $xy \in \mathfrak{b}$.

- Also, it is closed under addition since $\overline{x}+\overline{y} \in \phi(\mathfrak{b})$ implies $x, y \in \mathfrak{b}$ implies $x+y \in \mathfrak{b}$ implies $\overline{x+y} \in \phi(\mathfrak{b})$.

From this we induce a map from Λ to Δ. WTS it is 1-1 and order preserving.

- Suppose $\exists \mathfrak{b}, \mathfrak{b}' \in \Lambda$ such that $\phi(\mathfrak{b}) = \phi(\mathfrak{b}')$. If $\mathfrak{b} \neq \mathfrak{b}'$, then without loss of generality, $\exists x \in \mathfrak{b}$ such that $x \notin \mathfrak{b}'$. This implies $\phi(x) \in \phi(\mathfrak{b}) = \phi(\mathfrak{b}')$, thus $\exists y \in \mathfrak{b}'$ such that $\phi(x) = \phi(y)$, $x \neq y$. Thus,

  $$\phi(y-x) = 0 \implies y - x \in \mathfrak{a} \implies y - x = a \in \mathfrak{a}$$

  for some a, thus

  $$x = y + a.$$

  Since $y \in \mathfrak{b}', a \in \mathfrak{a} \subseteq \mathfrak{b}'$, so $x \in \mathfrak{b}'$,contradiction. Hence ϕ is 1-1 map on Λ.

- If $\mathfrak{b} \supseteq \mathfrak{b}'$, then $\forall x \in \mathfrak{b}' \subseteq \mathfrak{b}$, $\phi(x) \in \phi(\mathfrak{b}) \implies \phi(\mathfrak{b}') \subseteq \phi(\mathfrak{b})$. Thus ϕ is order preserving.

Also note that $\phi^{-1}(\overline{\mathfrak{b}})$ is an ideal containing $\mathfrak{a}$; since $\ker \phi = \mathfrak{a}$, and if $x, y \in \phi^{-1}(\overline{\mathfrak{b}})$, $\phi(x+y) = \phi(x) + \phi(y) \in \overline{\mathfrak{b}} \implies x + y \in \phi^{-1}(\overline{\mathfrak{b}})$, and if $x \in A, y \in \phi^{-1}(\overline{\mathfrak{b}})$, then $\phi(xy) = \phi(x)\phi(y) \in \overline{\mathfrak{b}} \implies xy \in \phi^{-1}(\overline{\mathfrak{b}})$. □

Statement in p.3. *$f : A \to B$ ring homomorphism, $\mathfrak{q}$ be a prime ideal of B. Then, $f^{-1}(\mathfrak{q})$ is prime in A.*

Proof. Let $\bar{f} : A/f^{-1}(\mathfrak{q}) \to B/\mathfrak{q}$ in a natural way. Then it is well-defined since $\bar{x} = \bar{y}$ in $A/f^{-1}(\mathfrak{q})$ if and only if $x = a + y$ for some $a \in f^{-1}(\mathfrak{q})$ if and only if $f(x) = f(a) + f(y)$ in B if and only if $\bar{f}(\bar{x}) = \bar{f}(\bar{y})$. Also, it is injective since $\bar{f}(\overline{x}) = \bar{0}$ implies $f(x) \in \mathfrak{q}$ implies $x \in f^{-1}(\mathfrak{q})$, thus $\bar{x} = 0$. Hence $\mathrm{Im}(\bar{f})$ is a subring of $B/\mathfrak{q}$, which is integral domain, thus image is also integral domain, hence $A/f^{-1}(q)$ is integral domain, thus $f^{-1}(q)$ is prime. □

Statement in p.4. *Let $\{\mathfrak{a}_\alpha\}_{\alpha \in I}$ is a chain, i.e. any two elements has inclusion relationship. Let $\mathfrak{a} = \bigcup_\alpha \mathfrak{a}_\alpha$. Then, $\mathfrak{a}$ is ideal.*

Proof.

- Let $x, y \in \mathfrak{a}$. Then, $x \in \mathfrak{a}_\alpha, y \in \mathfrak{a}_\beta$ for some $\alpha, \beta \in I$. From chain conditoin, either $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ or $\mathfrak{a}_\alpha \supseteq \mathfrak{a}_\beta$. Without loss of generality, assume $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$. Then, $x + y \in \mathfrak{a}_\beta \subseteq \mathfrak{a}$.
- If $x \in A, y \in \mathfrak{a}$. Then, $y \in \mathfrak{a}_\alpha$ for some α, thus $xy \in \mathfrak{a}_\alpha \subseteq \mathfrak{a}$.

□

Corollary 1.4. *If $\mathfrak{a} \neq (1)$ is an ideal of A, there exists a maximal ideal of A containing $\mathfrak{a}$.*

Proof. $A/\mathfrak{a}$ has maximal ideal by Theorem 1.3, say $\overline{\mathfrak{m}}$. Then, if we let $\phi : A \to A/\mathfrak{a}$ as canonical projection, then $\mathfrak{m} := \phi^{-1}(\overline{\mathfrak{m}})$ is an ideal in A containing $\mathfrak{a}$, from proposition 1.1. If it is not maximal, then $\exists \mathfrak{b}$ such that $\mathfrak{m} \subsetneq \mathfrak{b} \subsetneq A$. By proposition 1.1, $\phi(\mathfrak{b})$ is a proper ideal of $A/\mathfrak{a}$ containing $\overline{\mathfrak{m}}$, contradicting maximality of $\overline{\mathfrak{m}}$. □

Corollary 1.5. *Every non-unit of A is contained in a maximal ideal.*

Proof. Let x be a nonunit. Then (x) is a proper ideal of A, apply corollary 1.4. □

Principal ideal is an ideal generated by 1 element. f is an ***nilpotent element*** if $\exists n \in \mathbb{N}$ such that $f^n = 0$.

Proposition 1.11. *1) Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be prime ideals and let $\mathfrak{a}$ be an ideal comtained in $\cup_{i=1}^n \mathfrak{p}_i$. Then, $a \subseteq \mathfrak{p}_i$ for some i.*

Proof. Its contrapositive form is

$$\mathfrak{a} \not\subseteq \mathfrak{p}_i \text{ for all } i \in [n] \implies \mathfrak{a} \not\subseteq \bigcup_{i=1}^n \mathfrak{p}_i.$$

So if $n = 1$, then the RHS of the contrapositive form is equal to LHS, done. Suppose it is true for $n - 1$, and supposet that $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ for all $i \in [n]$. Then, for each i, we have

$$\mathfrak{a} \not\subseteq \mathfrak{p}_j \text{ for } j \in [n] \setminus \{i\} \implies \mathfrak{a} \not\subseteq \bigcup_{j \neq i}^n \mathfrak{p}_j.$$

Hence, there exist $x_i \in \mathfrak{a}$ such that $x_i \notin \bigcup_{j \neq i}^n \mathfrak{p}_j$. If $x_i \notin \mathfrak{p}_i$ for some $i \in [n]$, then $x_i \notin \bigcup_{i=1}^n \mathfrak{p}_i$, so done. Otherwise, $x_i \in \mathfrak{p}_i$ for all i. Let

$$y = \sum_{i=1}^n \prod_{j \neq i}^n x_i.$$

Then, $y \in \mathfrak{a}$ but $y \notin \mathfrak{p}_i$ since each monomial is not in any $\mathfrak{p}_i$. Done. □

Exercise 1.12. *1.* $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$

2. $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$

3. $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$

4. $(\cap_i \mathfrak{a}_i : \mathfrak{b}) = \cap_i (\mathfrak{a}_i : \mathfrak{b})$.

5. $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \cap_i (\mathfrak{a} : \mathfrak{b}_i)$.

Proof. 1. $\forall x \in \mathfrak{a}$, $x\mathfrak{b} \subseteq \mathfrak{a}$, since ideal is multiplicatively closed.

2. Let $x \in (\mathfrak{a} : \mathfrak{b})$. Then $x\mathfrak{b} \subseteq \mathfrak{a}$ by definition of $(\mathfrak{a} : \mathfrak{b})$. Thus, $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.

3. Let $x \in ((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c})$. Then, $x\mathfrak{c} \subseteq (\mathfrak{a} : \mathfrak{b})$. So

$$x\mathfrak{b}\mathfrak{c} = x\mathfrak{c}\mathfrak{b} \subseteq \mathfrak{a}$$

where first equality comes from commutativity and the inclusion comes from $x\mathfrak{c} \subseteq (\mathfrak{a} : \mathfrak{b})$ which implies $x \in ((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c})$. The inclusion gives $x \in (\mathfrak{a} : \mathfrak{b}\mathfrak{c})$.

Also, if $x \in (\mathfrak{a} : \mathfrak{b}\mathfrak{c})$, then for any $y \in \mathfrak{c}$,

$$xy\mathfrak{b} \subseteq x\mathfrak{c}\mathfrak{b} = x\mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a}.$$

Thus $x\mathfrak{c} \subseteq (\mathfrak{a} : \mathfrak{b})$, which implies $x \in ((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c})$. And $x \in ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$ is clear from

$$(x\mathfrak{b})\mathfrak{c} \subseteq \mathfrak{a}.$$

Lastly, let $x \in ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$. Then,

$$(x\mathfrak{b})\mathfrak{c} \subseteq \mathfrak{a}$$

holds, so $x \in (\mathfrak{a} : \mathfrak{b}\mathfrak{c})$, which implies $x \in ((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c})$ by above argument.

4. Let $x \in (\cap_i \mathfrak{a}_i : \mathfrak{b})$. which is equivalent to $x\mathfrak{b} \subseteq \mathfrak{a}_i$ for all i, which is equivalent to $x \in \cap_i (\mathfrak{a}_i : \mathfrak{b})$.

5. $x \in (\mathfrak{a} : \sum_i \mathfrak{b}_i)$ implies

$$x\mathfrak{b}_i \subseteq x \sum_i \mathfrak{b}_i \subseteq \mathfrak{a}$$

which implies $x \in \cap_i (\mathfrak{a} : \mathfrak{b}_i)$. Conversely, $x \in \cap_i (\mathfrak{a} : \mathfrak{b}_i)$ implies $x \in (\mathfrak{a} : \mathfrak{b}_i)$ for all i, which implies $x \sum_i \mathfrak{b}_i \subseteq \mathfrak{a}$, done.

□

Exercise 1.13. *1.* $r(\mathfrak{a}) \supseteq \mathfrak{a}$

2. $r(r(\mathfrak{a})) = r(\mathfrak{a})$.

3. $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.

4. $r(\mathfrak{a}) = (1) \iff \mathfrak{a} = (1)$.

5. $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$

6. *if* $\mathfrak{p}$ *is prime, then* $r(\mathfrak{p}^n) = \mathfrak{p}$ *for all* $n > 0$.

Proof. 1. If $x \in \mathfrak{a}$, then $x^1 \in \mathfrak{a}$, thus done.

2. It suffices to show that $r(r(\mathfrak{a})) \subseteq r(\mathfrak{a})$. Let $x \in r(r(\mathfrak{a}))$. Then, $x^n \in r(\mathfrak{a})$ for some $n > 0$, thus $(x^n)^m \in \mathfrak{a}$ for some $m > 0$, which implies $x \in r(\mathfrak{a})$ since $x^{mn} \in \mathfrak{a}$.

3. From $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$, we know $r(\mathfrak{ab}) \subseteq r(\mathfrak{a} \cap \mathfrak{b})$. Also, if $x \in r(\mathfrak{a} \cap \mathfrak{b})$, then $\exists n \in \mathbb{N}$ such that $x^n \in \mathfrak{a} \cap \mathfrak{b}$, which implies $x \in r(\mathfrak{a}) \cap r(\mathfrak{b})$. Also, if $x \in r(\mathfrak{a}) \cap r(\mathfrak{b})$, then $x^n \in \mathfrak{a}, x^m \in \mathfrak{b}$ for some $n, m > 0$, thus $x^n x^m \in \mathfrak{ab}$, hence $x \in r(\mathfrak{ab})$. This implies

$$r(\mathfrak{ab}) \subseteq r(\mathfrak{a} \cap \mathfrak{b}) \subseteq r(\mathfrak{a}) \cap r(\mathfrak{b}) \subseteq r(\mathfrak{ab})$$

done.

4. if $r(\mathfrak{a}) = (1)$, then $1^n = 1 \in \mathfrak{a}$, so $\mathfrak{a} = (1)$. Conversely, use $r(\mathfrak{a}) \supseteq \mathfrak{a}$.

5. From first one we know $\mathfrak{a} \subseteq r(\mathfrak{a}), \mathfrak{b} \subseteq r(\mathfrak{b})$, thus

$$\mathfrak{a} + \mathfrak{b} \subseteq r(\mathfrak{a}) + r(\mathfrak{b})$$

which implies $x^n \in \mathfrak{a} + \mathfrak{b} \subseteq r(\mathfrak{a}) + r(\mathfrak{b})$ implies $x^n \in r(r(\mathfrak{a}) + r(\mathfrak{b}))$. Conversely, if $x \in r(r(\mathfrak{a}) + r(\mathfrak{b}))$, then there exists $n \in \mathbb{N}$ such that $x^n \in r(\mathfrak{a}) + r(\mathfrak{b})$. This implies $x^n = ca + db$ for some $c, d \in A, a \in r(\mathfrak{a}), b \in r(\mathfrak{b})$, thus $\exists n_a, n_b \in \mathbb{N}$ such that $a^{n_a} \in \mathfrak{a}, b^{n_b} \in \mathfrak{b}$. Hence, by investigating terms in binomial expansion of $(x^n)^{n_a+n_b+1}$, we can conclude, $(x^n)^{n_a+n_b+1} \in r(\mathfrak{a} + \mathfrak{b})$.

6. Let $x \in r(\mathfrak{p}^n)$ implies $x^k \in \mathfrak{p}^n$ for some $k > 0$, this implies x or x^{k-1} is in $\mathfrak{p}$, do the same thing at most $k-1$ time, we can conclude that $x \in \mathfrak{p}$. Thus, $r(\mathfrak{p}^n) \subseteq \mathfrak{p}$. And since any n-th power of element in $\mathfrak{p}$ should be in $\mathfrak{p}^n$, so $r(\mathfrak{p}^n) \supseteq \mathfrak{p}$, done.

□

Proposition 1.14. *The radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals which contain $\mathfrak{a}$.*

Proof. We already know $r(a) = \phi^{-1}(\mathfrak{R}_{A/\mathfrak{a}})$, where $\mathfrak{R}$ is nilradical, and $\phi : A \to A/\mathfrak{a}$ a canonical projection. Since $\mathfrak{R}_{A/\mathfrak{a}}$ is the intersection of all prime ideals in $A/\mathfrak{a}$, from the one-to-one correspondence and theorem 1.8 (preimage of prime ideal is prime), $r(a)$ is intersection of prime ideals containing $\mathfrak{a}$. □

Proposition 1.15 (part of proof). $D = r(D) = \bigcup_{x \neq 0} Ann(x)$

Proof. $D \subseteq r(D)$ is clear from above exercise. If $f \in r(D)$, then $f^n \in D$, so $f^n h = 0$ for some $h \in A$, so $f(f^{n-1}h) = 0$ implies $f \in D$.

Also, $f \in D$ implies $fx = 0$ for some $x \in A \setminus \{0\}$ implies $f \in Ann(x)$. Conversely, if $f \in Ann(x)$, then $f \in D$. Since x was chosen arbitrarily, done. □

Proposition 1.17. *$f : A \to B$.*

1. $\mathfrak{a} \subseteq \mathfrak{a}^{ec}, \mathfrak{b} \supseteq \mathfrak{b}^{ce}$.

2. $\mathfrak{a}^e = \mathfrak{a}^{ece}, \mathfrak{b}^c = \mathfrak{b}^{cec}$.

Proof. 1) Since $f(\mathfrak{a}) \subseteq \mathfrak{a}^e$,

$$\mathfrak{a} \subseteq f^{-1}(f(\mathfrak{a})) \subseteq f^{-1}(\mathfrak{a}^e) = \mathfrak{a}^{ec}.$$

Also, since $f(f^{-1}(\mathfrak{b})) \subseteq \mathfrak{b}$ from multiplicative closedness of ideal, $f(f^{-1}(\mathfrak{b}))^e \subseteq \mathfrak{b}$.

2)

$$\mathfrak{b}^c \subseteq (\mathfrak{b}^c)^{ec}$$

by 1) and $\mathfrak{b} \supseteq \mathfrak{b}^{ce} \implies \mathfrak{b}^c \supseteq \mathfrak{b}^{cec}$ since f^{-1} preserve order of ideal. Similarly,

$$\mathfrak{a}^e \supseteq (\mathfrak{a}^e)^{ce}$$

by 1) and $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ implies $f(\mathfrak{a}) \subseteq f(\mathfrak{a}^{ec})$ which implies $\mathfrak{a}^e \subseteq \mathfrak{a}^{ece}$. □

Exercise 1.18. *1.* $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$, *2.* $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$,

3. $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$

4. $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$,

5. $(\mathfrak{a}_1\mathfrak{a}_2)^e = \mathfrak{a}_1^e\mathfrak{a}_2^e$,

6. $(\mathfrak{b}_1\mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c\mathfrak{b}_2^c$,

7. $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e : \mathfrak{a}_2^e$,

8. $(\mathfrak{b}_1 : \mathfrak{b}_2)^c = \mathfrak{b}_1^c\mathfrak{b}_2^c$,

9. $r(\mathfrak{a})^e \subseteq r(\mathfrak{a}^e)$

10. $r(\mathfrak{b})^c = r(\mathfrak{b}^c)$.

The set of ideals E is closed under sum and product, and C is closed under the other three operations.

Proof. 1. $y \in (\mathfrak{a}_1 + \mathfrak{a}_2)^e \iff y = \sum c_i f(\alpha_i) + \sum d_j f(\beta_j) \in \mathfrak{a}_1^e + \mathfrak{a}_2^e$

2. $x \in \mathfrak{b}_1^c + \mathfrak{b}_2^c$ implies $x = cx_1 + dx_2$, thus $f(x) = cf(x_1) + df(x_2) \in \mathfrak{b}_1 + \mathfrak{b}_2$ implies $x \in f^{-1}(f(x)) \subseteq f^{-1}(\mathfrak{b}_1 + \mathfrak{b}_2) = (\mathfrak{b}_1 + \mathfrak{b}_2)^c$.

3. $y \in (\mathfrak{a}_1 \cap \mathfrak{a}_2)^e$ implies $y = \sum c_i f(x_i)$ with $x_i \in \mathfrak{a}_1 \cap \mathfrak{a}_2$. Hence, $f(x_i) \in \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$, which implies $y \in \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$.

4. $x \in (\mathfrak{b}_1 \cap \mathfrak{b}_2)^c \iff f(x) \in \mathfrak{b}_1 \cap \mathfrak{b}_2 \iff f(x) \in \mathfrak{b}_1$ and $f(x) \in \mathfrak{b}_2 \iff x \in \mathfrak{b}_1^c, \mathfrak{b}_2^c \iff x \in \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$.

5. $y \in (\mathfrak{a}_1\mathfrak{a}_2)^e \iff y = \sum c_i f(x_i y_i)$ where $x_i \in \mathfrak{a}_1, y_i \in \mathfrak{a}_2 \iff y = \sum c_i f(x_i) f(y_i)$. Notes that $f(x_i)f(y_i) \in \mathfrak{a}_1^e\mathfrak{a}_2^e$, so it is equivalent to say that $y \in \mathfrak{a}_1^e\mathfrak{a}_2^e$.

6. $x \in \mathfrak{b}_1^c\mathfrak{b}_2^c \implies x = \sum_i c_i x_i y_i$ for some $x_i \in \mathfrak{b}_1^c, y_i \in \mathfrak{b}_2^c$ implies $f(x) = \sum_i c_i f(x_i) f(y_i) \in \mathfrak{b}_1\mathfrak{b}_2$, thus $x \in f^{-1}(f(x)) \subseteq (\mathfrak{b}_1\mathfrak{b}_2)^c$.

7. $y \in (\mathfrak{a}_1 : \mathfrak{a}_2)^e$ implies $y = \sum c_i f(x_i)$ for some $x_i \in (\mathfrak{a}_1 : \mathfrak{a}_2)$. Then, $x_i\mathfrak{a}_2 \subseteq \mathfrak{a}_1$ implies $f(x_i)f(\mathfrak{a}_2) = f(x_i\mathfrak{a}_2) \subseteq f(\mathfrak{a}_1)$ implies $f(x_i)\mathfrak{a}_2^e \subseteq \mathfrak{a}_1^e$ since $(\cdot)^e$ is just idealization, so inclusion of generating set implies inclusion of ideal.

8. $x \in (\mathfrak{b}_1 : \mathfrak{b}_2)^c \implies f(x)\mathfrak{b}_2 \subseteq \mathfrak{b}_1 \implies f^{-1}(f(x))\mathfrak{b}_2^c \subseteq \mathfrak{b}_1^c \implies x\mathfrak{b}_2^c \subseteq \mathfrak{b}_1^c \implies x \in (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$. The other way is not true since $f(\mathfrak{b}_1^c)$ may not be $\mathfrak{b}_1$.

9. $y \in r(\mathfrak{a})^e \implies y = \sum_{i=1}^{l} c_i f(x_i)$ where $x_i^{m_i} \in \mathfrak{a}$ for some $m_i \in \mathbb{N}$, thus all terms in the expansion of $y^{\sum_i m_i + 1}$ should contain $x_i^{m_i}$, which implies $y^{\sum_i m_i + 1} \in \mathfrak{a}^e \implies y \in r(\mathfrak{a}^e)$.

10. $x \in r(\mathfrak{b})^c \iff f(x) \in r(\mathfrak{b}) \iff f(x)^m \in \mathfrak{b}$ for some $m \in \mathbb{N} \iff f(x^m) \in \mathfrak{b} \iff x^m \in \mathfrak{b}^c \iff x \in r(\mathfrak{b}^c)$.

□

1.1 Exercises in Section 1

1. Let x be a nilpotent element. Then $x^n = 0$ for some $n \in \mathbb{N}$. Then

$$(1+x)(1 - x + x^2 - \cdots + (-1)^n x^n) = 1 + (-1)^n x^{n+1} = 1$$

Let u be a unit. Then, $u^{-1}x$ is still nilpotent, since $(u^{-1}x)^n = u^{-n}x^n = 0$. Thus $1 + u^{-1}x$ is unit, thus $u + x$ is unit.

2. (a) If f is unit, then $fg = 1$ for some $g = b_0 + b_1x + \cdots + b_mx^m$. (Without loss of generality, assume $n > m$.) Thus,

$$\begin{aligned} a_nb_m &= 0 \\ a_{n-1}b_m + a_nb_{m-1} &= 0 \implies & a_na_{n-1}b_m + a_n^2b_{m-1} &= 0 \implies & a_n^2b_{m-1} = 0 \\ a_{n-2}b_m + a_{n-1}b_{m-1} + a_nb_{m-2} &= 0 \implies & a_n^2a_{n-2}b_m + a_n^2a_{n-1}b_{m-1} + a_n^3b_{m-2} &= 0 \implies & a_n^3b_{m-2} = 0 \\ &\cdots \end{aligned}$$

So if $a_n^{r+1}b_{m-r} = 0$ for some $r = s-1$, then for $r = s$, the coefficient of fg with degree s is

$$a_{n-s}b_m + a_{n-s+1}b_{m-1} + \cdots + a_nb_{m-s} = a_n^sa_{n-s}b_m + a_n^sa_{n-s+1}b_{m-1} + \cdots + a_n^{s+1}b_{m-s} = 0$$

and by inductive hypothesis, $a_nb_m = 0, a_n^2b_{m-1} = 0, \cdots, a_n^sb_{m-s+1} = 0$ implies $a_n^{s+1}b_{m-s} = 0$, done. Hence, we can say that

$$a_n^{m+1}b_0 = 0$$

Since b_0 is unit (since $a_1b_1 = 1$ in the $fg = 1$) it means $a_n^{m+1} = 0$ so a_n is nilpotent, which implies $-a_nx^n$ is nilpotent (since $(-a_nx^n)^{m+1} = (-1)^{m+1}a_n^{m+1}x^{m+n+1} = 0$,) thus $f - a_nx^n$ is nilpotent since it is sum of unit and nilpotent element, which is nilpotent by exercise 1. Thus, by applying the same argument on $f - a_nx^n$, we get a_{n-1} is nilpotent, and applying it $n-1$ times, we get $a_n, a_{n-1}, \cdots, a_1$ are nilpotent, done.

Conversely, if $f = a_0 + a_1x + \cdots + a_nx^n$ is an element in $A[x]$ such that a_0 is a unit, and $a_1, \cdots, a_n$ are nilpotent. This implies $a_ix^i, i = 1, \cdots, n$ are nilpotent, thus $f - a_0$ is nilpotent, since set of all nilpotent element is an ideal, thus it is closed under addition. Then, f is sum of unit and nilpotent element, which is a unit by the exercise 1.

Note that there is more abstract version of the answer for if direction. Let $\mathfrak{p}$ be any prime ideal. Then, $A/\mathfrak{p}$ is an integral domain, and we know that $\overline{f} \in (A/\mathfrak{p})[x]$ is a unit if and only if $\overline{f} = u$ for some unit in $A/\mathfrak{p}$; to see this, otherwise, suppose $\overline{f} = \overline{a_n}x^n + \cdots + \overline{a_1}x + \overline{a_0}$ with $a_n \neq 0$ is a unit in $A/\mathfrak{p}[x]$. Then, there exists $\overline{g} = \overline{b_0} + \overline{b_1}x + \cdots + \overline{b_m}x^m$ with $\overline{b_m} \neq 0$ for $m > 0$ such that $\overline{f}\overline{g} = 1$ implies $\overline{b_m} = 0$, contradiction. Thus inverse element of $\overline{f}$ should be in $A/\mathfrak{p}$, and in this case, $\overline{f}^{-1}\overline{a_n} \neq 0$ if $\overline{f}^{-1}$, so there is no such inverse element, contradiciton.

So, f is unit implies $\overline{f}$ is unit (from $1 = \phi(1) = \phi(f \cdot f^{-1}) = \phi(f)\phi(f^{-1})$), which implies $\overline{f} = u$ for some $u \in A/\mathfrak{p}$, and $\mathfrak{p}$ is chosen arbitrarily implies a_0 is unit and $a_1, \cdots, a_n$ are in the intersection of all prime ideals, i.e., nilradical, which implies they are nilpotent.

(b) Suppose $a_0, \cdots, a_n$ is nilpotent. Then, a_ix^i is nilpotent by the same argument used in the above proof, so f is nilpotent. Conversely, if f is nilpotent, then $f^m = 0$ for some $m \in \mathbb{N}$, and f^m has the highest degree term $a_n^mx^{mn} = 0$, thus a_n is nilpotent. Since nilradrical is closed under addition, $f - a_nx^n$ is also nilpotent, and apply the same argument to conclude that a_{n-1} is nilpotent. Do the same argument on $f - a_nx^n - a_{n-1}x^{n-1}$, $\cdots$, so that we can get $a_n, \cdots, a_0$ are nilpotent.

Or, we can say that f is nilpotent, thus $\phi(f)$ is nilpotent for any prime ideal $\mathfrak{p}$ and $\phi : A[x] \to A/\mathfrak{p}[x]$. Since $A/\mathfrak{p}$ is integral domain, so does $A/\mathfrak{p}[x]$, hence $\phi(f) = 0$. This implies $a_0, \cdots, a_n \in \mathfrak{p}$. Since $\mathfrak{p}$ was arbitrarily chosen, $a_0, \cdots, a_n$ are nilpotent.

(c) only if part is just definition of zero divisor. So suppose f is zero divisor. Then, $\exists g = b_0 + b_1x + \cdots + b_mx^m$ such that $fg = 0$ and $\deg(g)$ is minimal for all other zero divisor of f. Then, $a_nb_m = 0$ thus $\deg(a_ng) = \deg(g) - 1$ and $f(a_ng) = a_nfg = 0$, thus $a_ng = 0$ from the minimality of g. Now suppose that $a_{n-s}g = 0$ for $s = 0, 1, \cdots, r-1$. Then,

$$fg = (a_0 + a_1x + \cdots + a_{n-s-1}x^{n-s-1})g + (a_{n-s}x^{n-s} + \cdots + a_nx^n)g = (a_0 + a_1x + \cdots + a_{n-s-1}x^{n-s-1})g$$

thus $a_{n-s-1}b_m = 0$, which implies $\deg(a_{n-s-1}g) = \deg(g) - 1$ and $a_{n-s-1}g$ is also zero divisor of f, thus $a_{n-s-1}g = 0$, which implies $a_{n-r}g = 0$. This implies $a_ib_m = 0$ for all $i \in [n] \cup \{0\}$, which implies $b_mf = 0$.

(d) Let $\mathfrak{a} = (a_0, \cdots, a_n), \mathfrak{b} = (b_0, \cdots, b_m)$, and let $\mathfrak{c} = (a_0b_0, a_1b_0 + a_0b_1, \cdots,)$ an ideal generated by coefficients of fg. Then the statement is equivalent to say that

$$\mathfrak{a} = (1) = \mathfrak{b} \iff \mathfrak{c} = (1)$$

Note that all generator of $\mathfrak{c}$ is in $\mathfrak{a}$ and $\mathfrak{b}$. Thus, $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$. Thus, if $\mathfrak{c} = (1)$ then $\mathfrak{a} = (1) = \mathfrak{b}$. Conversely, suppose $\mathfrak{a} = (1) = \mathfrak{b}$, but to get a contradiction, $\mathfrak{c} \neq (1)$. Then, take a maximal ideal $\mathfrak{m}$ containing $\mathfrak{c}$. Then, by sending fg from $A[x]$ to $A/\mathfrak{m}[x]$, $fg \mapsto 0$ since $\mathfrak{m}$ contains all coefficient of fg. However, f and g are not zero in $A/\mathfrak{m}[x]$, since if those are zero, then coefficients of f or those of g should be in $\mathfrak{m}$, contradicting the assumption that $\mathfrak{a} = (1) = \mathfrak{b}$. Thus f, g are zero divisor in $A/\mathfrak{m}[x]$. However, $A/\mathfrak{m}$ is a field, so $A/\mathfrak{m}[x]$ is an integral domain, which implies no zero divisors exist, contradiction.

3. (a) In multivariate case, use abstract argument; Let $\mathfrak{p}$ be any prime ideal. Then, $A/\mathfrak{p}$ is an integral domain, and we know that $\overline{f} \in A/\mathfrak{p}[x_1, \cdots, x_n]$ is a unit if and only if $\overline{f} = u$ for some unit in $A/\mathfrak{p}$; to see this, suppose f contains a term $c\vec{x}^{\vec{m}}$ with $\vec{m}$ is not a zero vector in $\mathbb{N}^n$ and $c \neq 0$. Then for any $\overline{g}$ with $g \neq 0$, let a highest degree term of g as $d\vec{x}^{\vec{m}'}$ and $d \neq 0$, then fg should contain $cd\vec{x}^{\vec{m}+\vec{m}'} = 0$, which implies $cd = 0$. However, $A/\mathfrak{p}$ is an integral domain, this implies either c or d is zero, contradiction. Thus, coefficients of f which is not a constant term should be in $\mathfrak{p}$, and the constant term should be unit. Since $\mathfrak{p}$ was arbitrarily chosen from the set of all prime ideals of A, and the intersection of prime ideal is nilradrical, we can conclude that all coefficients of f except constant term are nilpotent, and the constant term is a unit.

   Conversely, if f has a unit constant term and all nilpotent coefficients, then note that nilpotent coefficient times $\vec{x}^{\vec{m}}$ is also nilpotent, so f is sum of unit and nilpotent in $A[x]$, thus f is a unit by the exercise 1.

   (b) Still, the only if part is just derived from that nilradrical is an ideal, thus closed under addition. So suppose f is a nilpotent. Then, for any $\mathfrak{p}$, a prime ideal, $\overline{f}$ on $A/\mathfrak{p}[\vec{x}]$ should be zero, since f is nilpotent implies $f^n = 0$ for some $n \in \mathbb{N}$, therefore $\phi(f)^n = 0$ implies $\overline{f} = \phi(f) = 0$. Hence, all coefficients of f should be in $\mathfrak{p}$, and since $\mathfrak{p}$ is arbitrarily chosen, all coefficients of f should be in nilradrical.

   (c) Still, only if part is just satisfy definition of zero-divisor. Suppose f is zero divisor. We use induction. Note that the statement holds for $A[x_1]$. Now suppose the statement holds for $A[x_1, \cdots, x_{n-1}]$. Then, let $f \in A[x_1, \cdots, x_n]$ is a zero divisor. Since $A[x_1, \cdots, x_n] = A[x_1, \cdots, x_{n-1}][x_n]$ hence we can think of f as a polynomial of x_n having coefficients from $A[x_1, \cdots, x_{n-1}]$. So let

$$f = \sum_{i=0}^{m} c_i(x_1, \cdots, x_{n-1})x_n^i.$$

   Then from the inductive hypothesis, for each c_i, there exists $b_i \in A$ such that $b_ic_i = 0$. Thus let $b = \prod_{i=0}^{m}$. Then, $bf = 0$ since $bc_i = 0$ for all i.

   (d) Still, if we let $\mathfrak{a}, \mathfrak{b}$ be an ideal generated by coefficients of f and g respectively, and let $\mathfrak{c}$ be an ideal generated by coefficients of fg. Then by construction, $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$, thus $\mathfrak{c} = (1)$ implies $\mathfrak{b} = (1) = \mathfrak{a}$. Conversely, suppose $\mathfrak{b} = (1) = \mathfrak{a}$ but $\mathfrak{c} \neq (1)$. Then, by applying the same argument above, we can get a contradiction.

4. From the fact that every maximal ideal is prime ideal, the Jacobson radical in $A[x]$ should contain nilradrical. So suppose f is in the Jacobson radical. It suffices to show that f is nilpotent. From proposition 1.9, $1 + xf$ is a unit in $A[x]$. By the exercise 2, all nonzero coefficients of xf should be nilpotent. This implies all coefficients of f is nilpotent, hence f is nilpotent by the exercise 2, done.

5. (a) If $f(x)$ is unit in $A[[x]]$, then $\exists g \in A[[x]]$ such that $fg = 1$. Thus if we let $g = \sum_{i=0}^{\infty} b_ix^i$, then $a_0b_0 = 1$. Hence, a_0 is unit in A.

   Conversely, suppose a_0 is unit. Then notes that for any $g = \sum_{i=0}^{\infty} b_ix^i$,

$$fg = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_jb_{i-j} \right) x^i.$$

Thus it suffices to find some g such that $\sum_{j=0}^{i} a_j b_{i-j} = 0$ for all $i > 0$. Notes that

$$0 = \sum_{j=0}^{i} a_j b_{i-j} = a_0 b_i + \sum_{j=0}^{i-1} a_j b_{i-1} \implies b_i = a_0^{-1} \sum_{j=0}^{i-1} a_j b_{i-1}.$$

So recursively we can determine b_i from $b_0 = a_0^{-1}$ and the above equation.

(b) To do this, we claim

Claim I. *If R is integral domain, so does $R[[x]]$.*

Proof. If $f, g \in R[[x]]$ with nonzero coefficient, then let ax^n and bx^m are the smallest nonzero elements in supports of f and g respectively. Then $fg = abx^{n+m}+$ highest order terms, thus $fg \neq 0$. □

Suppose f is nilpotent. Then $f^n = 0$ for some $n \in \mathbb{N}$. Fix $\mathfrak{p}$ a prime ideal and $\phi : A[[x]] \to A/p[[x]]$. Then, $\phi(f)$ is zero since $\phi(f)$ should be nilpotent. Since $\mathfrak{p}$ was arbitrarily chosen, all coefficients of f are nilpotent.

Converse is not true in general. See [1].

(c) Suppose f is in the Jacobson radical of $A[[x]]$. This is equivalent to say that for any $g \in A[[x]]$, $1 - fg$ is unit, by Proposition 1.9. This is equivalent to say that $1 - a_0 c$ is unit for all $c \in A$ which is constant part of g, by exercise 5 i). This is equivalent to say that a_0 is in the Jacobson radical.

(d) Notes that $A = A[[x]]/(x)$. Hence $\phi : A[[x]] \to A$ is canonical projection, thus there is one-to-one correspondence from ideal of A and ideal of $A[[x]]$ containing (x). Now we claim that $x \in \mathfrak{R}$, the Jacobson radical of $A[[x]]$. This is because for any $f \in A[[x]]$, $1 - xf$ is unit by Exercise 5 i). Hence every maximal ideal $\mathfrak{m}$ of $A[[x]]$ contains x.

Also, we claim that if $\psi : A \to A[[x]]$ as canonical injection, then $\psi^{-1}(\mathfrak{m}) = \phi(\mathfrak{m})$. To see this, notes that for any $f \in A[[x]]$ with $f = a_0 + \sum_{i=1}^{\infty} a_i x^i$, $\phi(f) = a_0$ and $\psi^{-1}(f) = \begin{cases} a_0 & \text{if } f = a_0 \\ 0 & \text{o.w.} \end{cases}$.

This implies $\psi^{-1}(\mathfrak{m}) \subseteq \phi(\mathfrak{m})$. Now let $f \in \mathfrak{m}$, with $f = a_0 + \sum_{i=1}^{\infty} a_i x^i$. Then, $\sum_{i=1}^{\infty} a_i x^i \in (x) \subseteq \mathfrak{m}$ implies $a_0 \in \mathfrak{m}$, hence $a_0 \in \psi^{-1}(\mathfrak{m})$. This implies $\psi^{-1}(\mathfrak{m}) \supseteq \phi(\mathfrak{m})$, done.

Thus, from these claim, $\mathfrak{m}^c = \psi^{-1}(\mathfrak{m}) = \phi(\mathfrak{m})$, and since ϕ preserves order of ideals by inclusion, $\mathfrak{m}^c$ is still maximal. Also, $\mathfrak{m}$ contains $\mathfrak{m}^c$ by above construction, and also contains x, thus $\mathfrak{m} \supseteq (\mathfrak{m}^c, x)$. Conversely, if $f \in \mathfrak{m}$, then it has constant part and xg part for some $g \in A[[x]]$, thus $f \in (\mathfrak{m}^c, x)$. This shows $\mathfrak{m} = (\mathfrak{m}^c, x)$.

(e) Let $\mathfrak{p}$ be a prime ideal of A. We claim that $(\mathfrak{p}, x)$ is a prime ideal in $A[[x]]$. To see this, think about $A[[x]]/(\mathfrak{p}, x)$. If $\phi : A[[x]]] \to A[[x]]/(\mathfrak{p}, x)$, then for any $f = \sum_{i=0}^{\infty} a_i x^i$, $\phi(f) = \phi(a_0)$, and $\phi(a_0) = 0$ if and only if $a_0 \in \mathfrak{p}$. We claim that $A[[x]]/(\mathfrak{p}, x) \cong A/\mathfrak{p}$; notes that all elements in $A[[x]]/(\mathfrak{p}, x)$ can be denoted by $\phi(a_0)$ for some $a_0 \in A$, so take a map $\phi(a_0) \mapsto \overline{a_0}$. It is well-defined since if $\phi(a) = \phi(b)$ for some $a, b \in A$, then $a + f = b$ for some $f \in (\mathfrak{p}, x)$, thus $f \in \mathfrak{p}$, otherwise $a + f$ is not in A. This implies $\overline{a} = \overline{b}$. Also it is injective since $\overline{b} = 0$ implies $b \in \mathfrak{p}$, thus $\phi(b) = 0$. Also surjectivity is clear.

Hence, from this isomorphism, we know that $A[[x]]/(\mathfrak{p}, x)$ is integral domain. Thus $(\mathfrak{p}, x)$ is prime ideal. And contraction of $(\mathfrak{p}, x)$ is $\mathfrak{p}$, since $(\mathfrak{p}, x)^c = (\mathfrak{p}, x) \cap A = \mathfrak{p}$. ,

6. Let J be the Jacobson radical and $\mathfrak{R}$ be nilradical. Suppose $J \supsetneq \mathfrak{R}$. Then, it contains $e \in J \setminus \mathfrak{R}$ such that $e^2 = e$. Now $e(1 - e) = 0$, and $1 - e$ has a form $1 + xe$ for some $x = -1 \in A$, thus $1 - e$ is unit in A by Proposition 1.9. Hence, $e = 0$, contradiction. This implies $J = \mathfrak{R}$.

7. Let J be the Jacobson radical and $\mathfrak{R}$ be nilradical. Suppose $J \supsetneq \mathfrak{R}$. Then, it contains $x \in J \setminus \mathfrak{R}$ such that $x^n = x$ for some $n \in \mathbb{N}$. Now $x(1 - x^{n-1}) = 0$, and $1 - x^{n-1}$ has a form $1 + xy$ for some $y = -x^{n-1} \in A$, thus $1 - x^{n-1}$ is unit in A by Proposition 1.9. Hence, $x = 0$, contradiction. This implies $J = \mathfrak{R}$. It implies all prime ideals are maximal ideal, otherwise $J \neq \mathfrak{R}$.

8. Let σ be the set of all prime ideals. Then $0 \in \sigma$, so it is nonempty. Give inverse inclusion as an order. Then, a chain of prime ideals has maximal element, which is the intersection of all prime ideals in the chain. Actually, the stating that intersection is prime is not trivial; let $\{\mathfrak{p}_i\}_{i\in I}$ is such a chain for some index set I, and let $xy \in \bigcap_{i\in I} \mathfrak{p}_i$ but $x, y \notin \bigcap_{i\in I} \mathfrak{p}_i$. This implies that there exists $\mathfrak{p}_j, \mathfrak{p}_k$ such that $x \notin \mathfrak{p}_j, y \notin \mathfrak{p}_k$. From the chain condition, we can assume that $j < k$, thus $x, y \notin \mathfrak{p}_j$. This implies $xy \notin \mathfrak{p}_j$, thus $xy \notin \bigcap_{i\in I} \mathfrak{p}_i \subset \mathfrak{p}_j$, contradiction. Thus the intersection of all prime ideals in the chain is prime. Thus each chain has the upper bound. By Zorn's lemma, σ has a maximal element, which is a minimal element in the set of all prime ideals, i.e., Spec A.

9. If $\mathfrak{a} = r(\mathfrak{a})$, then by Proposition 1.14, done. If $\mathfrak{a}$ is intersection of prime ideals, say $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$, then $x^n \in \mathfrak{a}$ implies $x^n \in \mathfrak{p}_i$ for all i, thus $x \in \mathfrak{p}_i$ for all i, thus $x \in \bigcap_i \mathfrak{p}_i = \mathfrak{a}$. This implies $r(\mathfrak{a}) = \mathfrak{a}$.

10. i) implies ii): i) implies nilradical is the only one prime ideal, say $\mathfrak{p}$. Also it implies that Jacobson radical is equal to nilradical. Thus, if $x \in \mathfrak{p}$, then x is nilpotent; otherwise, $x \in A \setminus \mathfrak{p}$. If x is not a unit, then x is contained in some maximal ideal which is $\mathfrak{p}$, contradiction. Hence x is unit.

    ii) implies iii): notes that nilradical has all nilpotents elements, so outside of the nilradical, every elements is a unit. Apply Proposition 1.6 i) to get the nilradical is maximal ideal and A is local ring.

    iii) implies i): Notes that nilradical is maximal ideal. Thus if $\mathfrak{p}$ is prime ideal in A, then $\mathfrak{p}$ should be nilradical; thus there is only one prime ideal.

11. (a) $(-x) = (-x)^2 = x^2 = x$.

    (b) Let $\mathfrak{p}$ be an ideal that is not contained in the nilradical. Then it has a nonzero element which is not in nilradical. Also, such element is nonzero idempotent, since it is Boolean ring. By Exercise 6, nilradical is equal to the Jacobson radical, thus every prime ideal is maximal.

    (c) Let $\mathfrak{a} = (a, b)$ and $\mathfrak{c} = (a + b + ab)$. Then, $\mathfrak{c} \subseteq \mathfrak{a}$. Now notes that $a(a + b + ab) = a^2 + ab + a^2b = a + 2ab = a$, and $b(a + b + ab) = b^2 + ab + ab^2 = b + 2ab = b$. Hence $\mathfrak{c} = \mathfrak{a}$. Now use induction; let $\mathfrak{a} = (a_1, \cdots, a_n)$ and suppose that any ideal generated by at most $n - 1$ elements are principal. Then, $\mathfrak{a} = (a_1 + a_2 + a_1a_2, a_3, \cdots, a_n)$ by the same argument. Now apply inductive hypothesis.

12. If the local ring contains a nonzero idempotent except 1, say e, then e is nonunit, since it is zero divisor, i.e., $e(e - 1) = 0$; so if it is unit, then $e^{-1}e(e - 1) = 0 \implies e = 1$, contradiction. Thus by Corollary 1.5, it is contained in a maximal ideal. Since A is local ring, it is contained in the maximal ideal, which is also a Jacobson radical. However, this implies that $1 - e$ is unit, since $1 - e$ has a form $1 + xe$ with $x = -1$, and Proposition 1.9 says that element with such form is unit. This also gives us $e(e - 1)(e - 1)^{-1} = 0$ implies $e = 0$, contradiction.

13. We use Proposition V. §2, 2.3. in [2], which stating that for $f \in K[x]$ where k is a field, there exists a field extension such that f has a root.

    If $\alpha = (1)$, then

$$1 = \sum_{i=1}^{n} f_i(x_{f_i})g_{f_i}$$

    for some $f_i \in K[x]$ and $g_{f_i} \in A$. Since each g_{f_i} are also polynomials, they are in some polynomial ring with finite number of variables. So we can assume that $g_{f_i} = g_{f_i}(x_{f_1}, \cdots, x_{f_N})$ for some $f_{n+1}, \cdots, f_N \in K[x]$. Just write f_i to i for simplicity. Then,

$$1 = \sum_{i=1}^{n} f_i(x_i)g_i(x_1, \cdots, x_N).$$

    Now by applying Proposition 2.3 in [2] n times, we can get a finite field extension such that $f_1, \cdots, f_n$ has a roots. Say roots of f_i be α_i. Also let $\alpha_i = 0$ if $n < i \leq N$. Then, replacing x_i with α_i gives $1 = 0$, contradiction.

    The rest step is to show that L is a field. If $a, b \in L$, then there exists n such that $a, b \in K_n$ so any operation on fields well-defined in L. Also, every $f \in \sigma$ has all of its roots in L by construction, so it splits into linear factors. Now take subset of L having all algebraic elements over K. Say it $\overline{K}$. Then definitely, every polynomial in $\overline{K}[x]$ splits into linear form, since each of this polynomial can be regarded as polynomial over $K_n[x]$ for some n. Thus it is algebraic closure of K.

14. Notes that $0 \in \sigma$, so it is nonempty. Now take a chain $\{\mathfrak{a}_i\}_{i \in I}$ for some index set I. Then, their union is also an ideal in which every element is a zero divisor; to see this, let x, y be in the union. Then, from total ordering, there exists $i \in I$ such that $x, y \in \mathfrak{a}_i$, hence their sum is also in $\mathfrak{a}_i$, done. Thus, by the Zorn's lemma, there exists a maximal element in Σ.

    Let $\mathfrak{m}$ be a maximal element of Σ. To see that it is a prime ideal, let $xy \in \mathfrak{m}$. Notes that xy is zero divisor, so there exists nonzero element a such that $xya = 0$. This implies x and y are zero divisors, since $x(ya) = 0$ and $y(xa) = 0$ with nonzero ya and xa respectively. Thus, $\mathfrak{m} + (x), \mathfrak{m} + (y)$ are an ideal in which every elements are zero divisors, but from maximality of $\mathfrak{m}$, $\mathfrak{m} + (x) = \mathfrak{m} = \mathfrak{m} + (y)$, which implies $x \in \mathfrak{m}$ or $y \in \mathfrak{m}$, hence $\mathfrak{m}$ is prime.

15. (a) From definition of $\mathfrak{a} = (E)$, any prime ideal containing E also contain $\mathfrak{a}$ and vice versa. Thus, $V(E) = V(\mathfrak{a})$. And any prime ideal containing $r(\mathfrak{a})$ also contain $\mathfrak{a}$. So it suffices to show that $V(\mathfrak{a}) \supseteq V(r(\mathfrak{a}))$. Let $\mathfrak{p} \in V(\mathfrak{a})$. If $x \in r(\mathfrak{a})$, then $x^n \in \mathfrak{a} \subseteq \mathfrak{p}$ for some n. This implies $x \in \mathfrak{p}$ from the property of prime ideal. Done.

    (b) $V(0) = X$ is clear; every ideal contains 0 as its subideal. $V(1) =$ is also clear; since no prime ideal contains 1 as its element.

    (c) Let $\mathfrak{p}$ contains $\bigcup_{i \in I} E_i$. Then it contains E_i for all I, thus $\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$. Conversely, let $\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$. Then it contains E_i for all I, done.

    (d) Follows from Exercise 1.13 ii) stating that $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$.

16. Notes that

$$\begin{aligned}
\operatorname{Spec}(\mathbb{Z}) &= \{(p) : p \text{ is prime in } \mathbb{Z} \text{ or } p = 0\} = \{(0), (2), (3), (5), \cdots\} \\
\operatorname{Spec}(\mathbb{R}) &= \{(0)\}
\end{aligned}$$

For $\operatorname{Spec}(\mathbb{C}[x])$, notes that $\mathbb{C}$ is a field, hence $\mathbb{C}[x]$ is a principal ideal domain. Thus, let (f) be a prime. If f is reducible, then $f = gh$ for some $g, h \in \mathbb{C}[x]$ thus (f) is not a prime; since f is polynomial with minimal degree in (f), and g and h has a degree less than f. Now if $\deg f = 0$, then $f = 0$, otherwise $(f) = (1)$, done. If $\deg f \geq 2$, then f splits into linear forms, since $\mathbb{C}$ is algebraically closed field. Thus, (f) is not prime. Hence only nonzero prime ideal occurs when f is linear form $(x - a)$. Thus,

$$\operatorname{Spec}(\mathbb{C}) = \{(x - a) : a \in \mathbb{C}\} \cup \{(0)\}.$$

We can think it as $\mathbb{C}$ with big points (0) whose (Zariski) closure is whole $\operatorname{Spec}(\mathbb{C})$.

For $\operatorname{Spec}(\mathbb{R}[x])$, also it is PID. Hence (f) is prime ideal if and only if f is irreducible. And their are two irreducible polynomials with respect to degree; f is linear form $x - a$, or f is degree 2 polynomial $f = x^2 + px + q$ such that $p^2 - 4q < 0$. Hence we can think it as a picture of $\mathbb{R}$ with big points (0), and $\{(p, q) \in \mathbb{R}^2 : p^2 - 4q < 0\}$.

For $\operatorname{Spec}(\mathbb{Z}[x])$, take two maps $\phi : \mathbb{Z} \to \mathbb{Z}[x]$ and $\psi : \mathbb{Z} \to \mathbb{Z}/(p)$ for each $p = 0$ or prime. Then, it gives us a below commuting diagram

$$\begin{tikzcd}
\operatorname{Spec}(k(p)[x]) \arrow[r] \arrow[d] & \operatorname{Spec}(\mathbb{Z}/(p)) \arrow[d, "\psi_*"] \\
\operatorname{Spec}(\mathbb{Z}[x]) \arrow[r, "\phi_*"] & \operatorname{Spec}(\mathbb{Z})
\end{tikzcd}$$

where $k(p)$ is a residue fields of (p). (To see that $\operatorname{Spec}(k(p)[x])$ is a pull back of two maps ϕ_* and ψ_*, if $p \neq 0$, then $k(p) = \mathbb{Z}_p$. Hence, there is a natural projection $\mathbb{Z}[x] \to \mathbb{Z}_p[x]$ and an injection $\mathbb{Z}_p \to \mathbb{Z}_p[x]$.

Thus from the map

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\psi} & \mathbb{Z}_p \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ \mathbb{Z}[x] & \longrightarrow & \mathbb{Z}_p[x] \end{array}$$

which are commutes and a pullback, we can get above pullback. (To see this concretely, let $\mathfrak{a}$ be an ideal in $\mathbb{Z}_p[x]$. Then, sending $\mathfrak{a}$ to $\mathbb{Z}_p$ gives $\mathfrak{a}\cap\mathbb{Z}_p$, which is 0 if $\mathfrak{a}$ contains no constant, or (1) if it has a nonzero constant. Also sending $\mathfrak{a}$ to $\mathbb{Z}[x]$ gives an ideal $\mathfrak{b}$ containing $\mathfrak{a}$ and containing (p). Notes that if $\mathfrak{a}$ contains a nonzero constant c, then $(p)+(c)$ is contained in $\mathfrak{b}$, thus $\mathfrak{b} = (1)$ since $(p)+(c) = 1$ by Bezout's theorem. Otherwise, $\mathfrak{b}\cap\mathbb{Z} = (p)$. So $\mathfrak{b}$ sends to an ideal containing (p) or $\mathbb{Z}$, and $\mathfrak{a}\cap\mathbb{Z}_p$ also sent to an ideal (p) or $\mathbb{Z}$. Thus $\mathrm{Spec}(\mathbb{Z}_p[x])$ satisfies the condition as a pullback of two maps ϕ_*, ψ_*.

Similarly, if $p = 0$, then $k(p) = \mathbb{Q}$, thus we can do the same analysis.

Hence, $\mathrm{Spec}(\mathbb{Z}[x]) = \{(p, f) : f \in \mathbb{Z}[x] \text{ and irreducible at } \mathbb{Z}_p[x]\}_{(p)\in\mathrm{Spec}\,\mathbb{Z}} \cup \{f \in \mathbb{Z}[x] : \text{ irreducible over } \mathbb{Q}[x]\}$. Notes that the first contains zero ideal and corresponding to pullback $\mathrm{Spec}(\mathbb{Z}_p[x])$. The second part corresponds to pullback $\mathbb{Q}[x]$, a fiber of the zero ideal.

17. To see X_f form a basis of open sets for Zariski topology, first of all, they cover $\mathrm{Spec}(A)$; just take f be a unit, then
$$X_f = V(f)^c = V(1)^c = \emptyset^c = \mathrm{Spec}(A).$$
Thus, we need to check that for any two open sets X_f and X_g has nonzero intersection and if $\mathfrak{p}$ is in the intersection, then there exists X_i containing h and contained in the intersection. To get this, notes that $\mathfrak{p}$ do not contain f and g. So take X_{fg}; then from $V(f), V(g) \subseteq V(fg)$, $X_f, X_g \supseteq X_{fg}$, which implies $X_f \cap X_g$ contains X_{fg}. Also, since $\mathfrak{p}$ do not contain f and g, neither does fg, otherwise from the prime ideal property, one of f and g should lie in $\mathfrak{p}$, contradiction. Hence $\mathfrak{p} \in X_{fg}$.

    (a) We already show that $X_{fg} \subseteq X_f \cap X_g$. Let $\mathfrak{p} \in X_{fg}$. Then, by the same argument, $\mathfrak{p}$ doesn't contain f and g, thus $\mathfrak{p} \in X_f \cap X_g$. This implies equality.

    (b) Suppose $X_f = \emptyset$. It is equivalent to say that every prime ideal contains f. It is equivalent to say that f is in the intersection of all prime ideals, i.e., nilradical.

    (c) $X_f = X$ is equivalent to say that no prime ideal contains f. This is equivalent to say that f is unit; (If f is unit then no prime ideal contains f. Conversely, if no prime ideal contains f but if f is nonunit, then (f) is proper ideal, thus by Corollary 1.5 there is some maximal ideal containing f, contradiction.)

    (d) $X_f = X_g$ if and only if $V(f) = V(g)$ if and only if $r(f) = r(g)$.

    (e) To see this, let $\{U_i\}_{i\in I}$ be an open cover of $\mathrm{Spec}(A)$. Then actually using basis we can find more finer open cover consisting of X_fs, say $\{X_f\}_{f\in J}$ for some index set j. This implies
    $$\mathrm{Spec}(A) = \bigcup_{f\in J} X_f = \bigcup_{f\in J} V(f)^c = (\bigcap_{f\in J} V(f))^c$$
    by De Morgan's law. Which implies
    $$\bigcap_{f\in J} V(f) = \emptyset.$$
    This implies that $(J) = (1)$. To see this, suppose not; then (J) is proper ideal, so by Corollary 1.5 there exists a maximal ideal containing (J), say $\mathfrak{m}$. Then $\mathfrak{m}$ is prime and contains all $f \in J$, which implies $\mathfrak{m} \in \bigcap_{f\in J} V(f) = \emptyset$, contradiction.
    Hence, $1 = \sum_{j=1}^n a_j f_j$ for some $f_j \in J$ and $a_j \in A$. This implies that
    $$\bigcap_{j=1}^n V(f_j) = V(\bigcup_{j=1}^n \{f_j\}) = \emptyset,$$

where first equality comes from Exercise 15 ii), and second equality comes from the fact that $1 = \sum_{j=1}^n a_j f_j$. This implies $\mathrm{Spec}(A) = \left(\bigcap_{j=1}^n V(f_j)\right)^c = \bigcup_{j=1}^n X_{f_j}$, which is finite subcover, done.

(f) To see that each X_f is quasi-compact, let $\{X_{f_i}\}_{i\in I}$ be an open cover of X_f. Then, $X_f = \bigcup_{i\in I} X_{f_i} \cap X_f = \bigcup_{i\in I} X_{ff_i}$. Thus for simplicity, replace ff_i with f_i so that assume $X_f = \bigcup_{i\in I} X_{f_i}$. This implies

$$V(f)^c = \bigcup_{i\in I} V(f_i)^c = (\bigcap_{i\in I} V(f_i))^c \implies V(f) = \bigcap_{i\in I} V(f_i) = V(\bigcup_{i\in I}\{f_i\})$$

where last equality comes from Exercise 15 ii). This implies that $r(f) = r(\{f_i\}_{i\in I}$, thus $f^n = \sum_{j\in J} a_j f_j$ for some finite subset J of I for some $n \in \mathbb{N}$. Thus, for any prime ideal $\mathfrak{p} \in V(\bigcup_{i\in I}\{f_i\})$, $f^n \in (\bigcup_{i\in J}\{f_i\})$ implies $f^n \in \mathfrak{p}$ thus $f \in \mathfrak{p}$ by prime property. Hence, $V(f) \supseteq V(\bigcup_{i\in J}\{f_i\})$. This implies

$$V(f)^c = X_f \subseteq V(\bigcup_{i\in J}\{f_i\})^c = \left(\bigcap_{i\in J} V(f_i)\right)^c = \bigcup_{i\in J} V(f_i)^c = \bigcup_{i\in J} X_{f_i}.$$

Thus $X_f \subseteq \bigcup_{i\in J} X_{f_i} \subseteq \bigcup_{i\in I} X_{f_i} = X_f$ implies $X_f = \bigcup_{i\in J} X_{f_i}$, so X_f has a finite subcover for any open conver. Hence it is quasi-compact.

(g) Let O be an open subset of X. If O is a finite union of sets X_f, say $O = \bigcup_{j\in J} X_{f_j}$ for some finite index set J, then finite union of compact set is compact. (Or say that each open cover of O is also an open cover of X_{f_j} for each $j \in J$, thus pick finitely subcover of X_{f_j} for each j and collect them.)

Conversely, suppose O is quasi-compact. then since $\{X_f\}$ is a basis of given topology and O is open, we can say $O = \bigcup_{i\in I} X_{f_i}$ for some (maybe infinite) index set I. Thus $\{X_{f_i}\}_{i\in I}$ is an open cover of O, hence we can take finite subset J of I such that $O \subseteq \bigcup_{i\in J} X_{f_i}$ from quasi-compactness. Hence

$$O \subseteq \bigcup_{i\in J} X_{f_i} \subseteq \bigcup_{i\in I} X_{f_i} = O$$

implies $O = \bigcup_{i\in J} X_{f_i}$, a finite union of sets X_f.

18. Prove ii) first. Then use ii) to prove i). Before begin with this problem, we just mention that from the exercise 15, the Zariski topology is precisely a topology whose closed sets are of form $V(E)$ for any subset E of A.

(a) If $\mathfrak{p}_x$ is maximal, then $V(\mathfrak{p}_x) = \{x\}$; if there exists a prime ideal containing maxima ideal $\mathfrak{p}_x$, then by maximality it should be $\mathfrak{p}_x$ itself. Hence, $\{x\}$ is closed. Conversely, let $\{x\}$ be closed. Then, $\{x\} = V(E)$ for some $E \subseteq A$. Then $\mathfrak{p}_x$ contains E. Also, notes that $x \in V(\mathfrak{p}_x) \subseteq V(E) = \{x\}$ implies $V(\mathfrak{p}_x) = \{x\}$. Hence if $\mathfrak{p}_x$ is not a maximal, then by Corollary 1.5, there is a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}_x$ properly, thus $\mathfrak{m} \in V(\mathfrak{p}_x)$, which implies $\mathfrak{m} = \mathfrak{p}_x$, contradiction. Hence $\mathfrak{p}_x$ is maximal.

(b) $V(\mathfrak{p}_x)$ contains x and is closed. Thus $V(\mathfrak{p}_x) \supseteq \overline{\{x\}}$. Conversely, $\overline{\{x\}} = V(E)$ for some subset E. Then $V(\mathfrak{p}_x) \subseteq V(E)$ implies $\mathfrak{p}_x \supseteq E$. Thus for any other prime ideal $\mathfrak{p}$ containing $\mathfrak{p}_x$, it contains E, thus $V(\mathfrak{p}_x) \subseteq V(E) = \overline{\{x\}}$, done.

(c) If $y \in \overline{\{x\}}$, then $y \in V(\mathfrak{p}_x)$ by ii), thus $\mathfrak{p}_y$ contains $\mathfrak{p}_x$. Conversely, if $\mathfrak{p}_y$ contains $\mathfrak{p}_x$, then $\mathfrak{p}_y \in V(\mathfrak{p}_x) = \overline{\{x\}}$ by ii).

(d) If x, y are distinct point of X, then either $\mathfrak{p}_x \subsetneq \mathfrak{p}_y$ or $\mathfrak{p}_y \subsetneq \mathfrak{p}_x$. Without loss of generality, assume $\mathfrak{p}_x \subsetneq \mathfrak{p}_y$. Then, $y \notin \overline{\{x\}}$. Thus, $X \setminus \overline{\{x\}}$ is an open set containing y but not x.

19. Suppose the nilradical is prime. Let O_1 and O_2 be two open sets. Then using basis we can take nonempty X_{f_1} and X_{f_2} which are subsets of O_1 and O_2 respectively. Then, f_1, f_2 are not nilpotent by Exercise 1.17 ii), so they are not in nilradical. Thus $f_1 f_2$ are not in the nilradical. Hence, $\emptyset \neq X_{f_1 f_2} = X_{f_1} \cap X_{f_2} \subseteq O_1 \cap O_2$, done.

Conversely, suppose X is irreducible. It suffices to show that for any f_1, f_2 which are not niilpotent, their product is also not nilpotent. From given not nilpotent condition, X_{f_1} and X_{f_2} are nonempty open sets by Exercise 1.17 ii). From the irreducibility condition, $X_{f_1} \cap X_{f_2}$ is also nonempty, thus $f_1 f_2$ is not nilpotent, done.

20. (a) Let U be a nonempty open set in $\overline{Y}$. Then, $U \cap Y$ is also open in Y by subspace topology, thus $U \cap Y$ is dense in Y as a subspace topology. Thus its closure is equal to closure of Y, which implies
$$\overline{Y} = \overline{U \cap Y} \subseteq \overline{U} = \overline{Y} \implies \overline{U} = \overline{Y},$$
done.

(b) Let Σ be a set of all irreducible subspace containing Y, which is an irreducible subspace of Y. Then, $Y \in \Sigma$, so it is nonempty. Give an ordering by inclusion, and take a chain. Then, union of all irreducible spaces, say $T = \bigcup_\alpha T_\alpha$ for some chain $\{T_\alpha\}$ irreducible; to see this, let U be an open set in T. Then, $U \cap T_\alpha$ is open in T_α for any α, thus closure of $U \cap T_\alpha$ is T_α for each α. Then,
$$T = \bigcup_\alpha T_\alpha = \bigcup \overline{U \cap T_\alpha} \subseteq \overline{U}$$
implies $\overline{U} = T$. Thus by the Zorn's lemma, Σ has a maximal element.

(c) Let Σ be a set of all maximal irreducible subspaces of X. First of all, it is closed since for any $T \in \Sigma$, its closure is also irreducible space, but by maximality, $T = \overline{T}$. Also, it covers X, since a singleton in X is irreducible as a subspace topology. (Notes that topology of singleton $\{x\}$ is $\{\emptyset, \{x\}\}$, thus only nonempty open set is $\{x\}$, which are definitely dense.) This implies that each singleton is contained in a maximal irreducible subspaces.

In a Hausdorff space, every irreducible space is singleton; to see this, let Y be any set strictly containing some singleton $\{x\}$. Then, for any $z \in Y \setminus \{x\}$, there exists two disjoint open sets U_z and U_x in X such that $z \in U_z, x \in U_x$. Thus, $U_z \cap Y$ and $U_x \cap Y$ are two open sets in Y (as a subspace topology) which are disjoint. Thus Y is not irreducible space. So every irreducible components of a Hausdorff space is singleton.

(d) We claim that every irreducible closed subset is exactly of form $V(\mathfrak{p})$ where $\mathfrak{p}$ is prime. If it holds, then for any irreducible space, its closure is of form $V(\mathfrak{p})$ from the Exercise 15 saying that every closed set in the Zariski topology is of form $V(E)$. Thus, the maximal irreducible set is just maximal irreducible closed subset, which is of form $V(\mathfrak{p})$ where $\mathfrak{p}$ is minimal prime; otherwise, there is a prime $\mathfrak{p}'$ which contained in $\mathfrak{p}$ thus $V(\mathfrak{p}') \supsetneq V(\mathfrak{p})$, hence not a maximal.

To see the claim, let $V(E)$ be an irreducible closed set. We can assume that $V(E) = V(I)$ for some radical ideal I generated by E. If I is not prime, let $a, b \notin I$ but $ab \in I$. Then $V((I,a)) \cup V((I,b)) = V((I,a) \cap (I,b)) = V((I,ab)) = V(I)$ from Exercise 15 iv). However, neither $V(I,a) = V(I)$ nor $V(I,b) = V(I)$. This implies $V(I)$ is not irreducible space.

Notes that I implicitly use the fact that

Claim II. *Y is irreducible space in X if and only if Y cannot be covered by two proper closed sets.*

Proof. To see this, Suppose Y cannot be covered by two proper closed sets. Let U and Vbe two arbitrary non-empty open sets of Y. Suppose U and Y has empty intersection, i.e., disjoint. Then, their complement are two proper closed subsets whose union is Y by De Morgan Law. Thus one of U^c or V^c contains Y, which contradicting the fact that U and V are two nonempty sets.

Conversely, suppose Y is irreducible. Let F, G are two proper closed sets whose union is Y. Then their complements are two open sets whose union is emptyset, by De Morgan Law, contradiction. $\square$

21. (a) Let $f \in A$. Then notes that for any $x \in Y_{\phi(f)} = V(\phi(f))^c$, which implies $\mathfrak{p}_x$ does not contain $\phi(f)$, thus $\phi^{-1}(\mathfrak{p}_x)$ does not contain f. (If $f \in \phi^{-1}(\mathfrak{p}_x)$, then $\phi(\phi^{-1}(x)) \subseteq \mathfrak{p}_x$ contains $\phi(f)$, contradiction.) Hence $\phi^{-1}(\mathfrak{p}_x) = \phi^*(x) \in X_f$. Thus, $Y_{\phi(f)} \subseteq (\phi^*)^{-1}(X_f)$.

Now let $y \in (\phi^*)^{-1}(X_f)$. Then, $\phi^*(y) \in X_f$. This implies $\phi^{-1}(\mathfrak{p}_y)$ does not contain f. Thus, $\phi(\phi^{-1}(\mathfrak{p}_y)) \subseteq y$ does not contain $\phi(f)$. This implies $\mathfrak{p}_y$ does not contain $\phi(f)$. (Otherwise, if $\phi(f) \in \mathfrak{p}_y$, then $\phi^{-1}(\mathfrak{p}_y)$ contain f, contradiction.) Hence $\mathfrak{p}_y \in Y_{\phi(f)}$.

(b) Let $x \in (\phi^*)^{-1}(V(\mathfrak{a}))$. Then, $\phi^{-1}(\mathfrak{p}_x) \in V(\mathfrak{a})$, thus $\mathfrak{a} \subseteq \phi^{-1}(\mathfrak{p}_x)$. Hence, $\phi(\mathfrak{a}) \subseteq \phi\left(\phi^{-1}(\mathfrak{p}_x)\right) \subseteq \mathfrak{p}_x$. Which implies $\mathfrak{a}^e \subseteq \mathfrak{p}_x$. Hence $x \in V(\mathfrak{a}^e)$.

Conversely, let $y \in V(\mathfrak{a}^e)$. Then $\mathfrak{p}_y$ contains $\mathfrak{a}^e$, which implies $\phi^{-1}(\mathfrak{p}_y)$ contains $\mathfrak{a}$, thus $\phi^{-1}(\mathfrak{p}_y) \in V(\mathfrak{a})$. Thus, $\phi^*(y) \in V(\mathfrak{a})$, which implies $y \in (\phi^*)^{-1}(V(\mathfrak{a}))$.

(c) Let $x \in \phi^*(V(b))$. Then there exists $y \in V(b)$ such that $\phi(y) = x$. This implies $\mathfrak{p}_x = \mathfrak{p}_y^c$. Since $\mathfrak{p}_y$ contains $\mathfrak{b}$ $\mathfrak{p}_x = \mathfrak{p}_y^c$ contains $\mathfrak{b}^c$. Thus $x \in V(\mathfrak{b}^c)$. From the closedness of $V(\mathfrak{b}^c)$,

$$\overline{\phi^*(V(b))} \subseteq V(\mathfrak{b}^c).$$

Conversely, from closedness of $\overline{\phi^*(V(b))}$, $\overline{\phi^*(V(b))} = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. Then, by ii),

$$V(\mathfrak{a}^e) = (\phi^*)^{-1}(V(\mathfrak{a})) = (\phi^*)^{-1}(\overline{\phi^*(V(b))}) \supseteq (\phi^*)^{-1}(\phi^*(V(b))) \supseteq V(b).$$

This implies

$$\mathfrak{a}^e \subseteq r(\mathfrak{b}).$$

Thus for any $f \in \mathfrak{a}$, $\phi(f) \in \mathfrak{a}^e \subseteq r(\mathfrak{b})$, thus $\phi(f^n) = \phi(f)^n \in \mathfrak{b}$ for some $n \in \mathbb{N}$, which implies $f^n \in \mathfrak{b}^c$. Hence $\mathfrak{a} \subseteq r(\mathfrak{b}^c)$. Thus

$$\overline{\phi^*(V(b))} = V(\mathfrak{a}) \supseteq V(r(\mathfrak{b}^c)) = V(\mathfrak{b}^c).$$

(d) First of all we claim that

Claim III. *III $\phi : A \to B$ is a ring isomorphism, then ϕ^* is homeomorphism.*

Proof. If ϕ is a ring isomorphism, then we have an inverse map $\psi : B \to A$. Now ϕ^* and ψ^* are inverse to each other, since $\phi^* \circ \psi^*(x) = \phi^{-1} \circ \psi^{-1}(x) = \psi \circ \phi(x) = x$. And they are continuous by i). Thus ϕ has an inverse map which is also continuous. Thus it is homeomorphism. $\square$

By isomorphism theorem, $A/\ker(\phi)$ is isomorphic to B as a ring. Hence, ϕ^* on B induces a homeomorphism between $\operatorname{Spec}(A/\ker(\phi))$ and $\operatorname{Spec}(B)$. (Notes that the given map sends open to open and closed to closed.) Now Proposition 1.1 implies that $\operatorname{Spec}(A/\ker(\phi))$ and $V(\ker(\phi))$ has order preserving 1-1 correspondence induced by the projection map. Thus it suffices to show that this map is homeomorphism. To see this, we can use the fact that bijective continuous map is homeomorphism iff it is open or closed map.

Let F be closed set in $A/\ker\phi$. Then $F = V(\bar{\mathfrak{a}})$ for some ideal $\bar{\mathfrak{a}}$ in $A/\ker(\phi)$ corresponding to an ideal $\mathfrak{a}$ in A by Proposition 1.1. Let $\pi : A \to A/\ker(\phi)$ be the canonical projection map. We claim that $\pi^*(F) = V(\mathfrak{a})$. If it holds, then π^* is bijective continuous and closed map, thus it is homeomorphism.

Let $y \in \pi^*(F)$. Then, $y = \pi^*(x)$ for some $x \in F$. This implies $\mathfrak{p}_y = \mathfrak{p}_x^c$. Also $\mathfrak{p}_x^c$ contains $\bar{\mathfrak{a}}^c = \mathfrak{a}$ since $\mathfrak{p}_x$ contains $\bar{\mathfrak{a}}$. This implies $\mathfrak{p}_y$ contains $\mathfrak{a}$, hence $y \in V(\mathfrak{a})$. Conversely, if $y \in V(\mathfrak{a})$, then $\mathfrak{p}_y$ contains $\mathfrak{a}$. Also, by Proposition 1.1 $\mathfrak{p}_y = \mathfrak{p}_x^c$ for some $\mathfrak{p}_x$ in $A/\ker(\phi)$. Hence, $\phi(\mathfrak{p}_y) = \mathfrak{p}_x$ and $\phi(\mathfrak{a}) = \bar{\mathfrak{a}}$. This implies $\mathfrak{p}_x$ contains $\bar{\mathfrak{a}}$. Thus $y = \pi^*(x)$ and from $x \in V(\bar{\mathfrak{a}})$ implies $y \in \pi^*(V(\bar{\mathfrak{a}})) = \pi^*(F)$. Hence π^* is closed map, thus done.

(e) From iii),

$$\overline{\phi^*(Y)} = \overline{\phi^*(V(0))} = V(0^c) = V(\ker(\phi)).$$

Thus, $\phi^*(Y)$ is dense if and only if $V(\ker(\phi)) = X$ if and only if $\ker(\phi) \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in X$ if and only if $\ker(\phi) \subseteq \mathfrak{R}$.

(f)

$$\psi \circ \phi^*(z) = (\psi \circ \phi)^{-1}(z) = \phi^{-1} \circ \psi^{-1}(z) = \phi^* \circ \psi^*(z).$$

(g) Notes that $\mathrm{Spec}(A) = \{\mathfrak{p}, 0\}$. Now think about $\mathrm{Spec}(B)$; notes that $A/\mathfrak{p}$ has only 0 as a prime ideal since it is a field, and so does K. We claim that

Claim IV. *IV Every ideal in direct product of (unital commutative) ring is direct product of two ideals in each ring.*

Proof. Let R, S be a two such rings, and let I be an ideal of $R \times S$. Then, I_R, a projection of I onto R is also ideal; to see this, let $r, r' \in I_R$. Then, $\exists s, s' \in S$ such that $(r, s), (r', s') \in I$, hence $(r + r', s + s') \in I$ therefore $r + r' \in I_R$. Also, for any $a \in R$ and $r \in I_r$, there exists $s \in S$ such that $(r, s) \in I$, hence for any $s' \in S$, $(a, s') \cdot (r, s) = (ar, s's) \in I$, thus $ar \in I_R$. We can do the same argument to show that I_S is also an ideal, thus $I = I_R \times I_S$. $\square$

Thus, since all ideals in $A/\mathfrak{p}$ is $(\bar{1})$ and $(\bar{0})$, and all ideals in K is K and (0). Hence B has four ideals, B, (0), and $\mathfrak{q}_1 = (\bar{0}) \times K$ and $\mathfrak{q}_2 = (\bar{1}) \times (0)$. Notes that $\mathfrak{q}_1 = \{(0, k) : k \in K\}, \mathfrak{q}_2 = \{(a, 0) : a \in A/\mathfrak{p}\}$.

Also, if $(a, k)(a', k') \in \mathfrak{q}_1$, then one of a or a' should be zero, this implies one of (a, k) o (a', k') lies in $\mathfrak{q}_1$. Thus $\mathfrak{q}_1$ is prime. By the similar argument, $\mathfrak{q}_2$ is prime. Notes that (0) is not a prime in B; since $(a, 0)(0, k) \in (0)$ but a, k are nonzero. Thus,

$$\mathrm{Spec}(B) = \{\mathfrak{q}_1, \mathfrak{q}_2\}.$$

Then, $\phi^{-1}(\mathfrak{q}_1) = \mathfrak{p}, \phi^{-1}(\mathfrak{q}_2) = 0$. Hence, ϕ^* is bijection. However, it is not homeomorphism. To see this, notes that $\mathrm{Spec}(A)$ has three closed sets, $\{0, \mathfrak{p}\}, \{\mathfrak{p}\}$, and $\emptyset$. $\{0\}$ is not a closed set, since $V(\mathfrak{p}_0) = V(0) = \mathrm{Spec}(A)$ since $\mathfrak{p}$ also contains 0. However, $\mathrm{Spec}(B)$ has discrete topology; to see this, notes that $V(\mathfrak{q}_1) = \{\mathfrak{q}_1\}, V(\mathfrak{q}_2) = \{\mathfrak{q}_2\}, V(\emptyset) = \mathrm{Spec}(B), V(B) = \emptyset$. Thus, $(\phi^*)(\{\mathfrak{q}_2\}) = \{0\}$ implies that ϕ^* is not a closed map, hence ϕ^* is not homeomorphism.

Notes that zero ideal is prime if the ring is integral domain.

22. (a) (First problem) Let $X_i := \{A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n : \mathfrak{p} \in \mathrm{Spec}(A_i)\}$. First of all, we claim that X_i as a subspace of $\mathrm{Spec}(A)$ has the topology homeomorphic to $\mathrm{Spec}(A_i)$. Before starting the proof, it is clear that $A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n$ is prime in A; if $(a, x, a')(b, y, b')$ is in the prime ideal, then either x or y should lie in $\mathfrak{p}$, thus either (a, x, a') or (b, y, b') lies in $A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n$, done.

To see this, let $\phi_i : A_i \to A$ by $x \mapsto (1, \cdots, 1, x, 1, \cdots, 1)$. Then, $\phi_i^* : \mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ is continuous map, thus $\phi_i^*|_{X_i} : X_i \to \mathrm{Spec}(A_i)$ is also continuous map. Now, notes that any closed set in X_i has a form $V(E) \cap X_i$ for some subset E of A. Hence $E = \prod_{j=1}^n E_j$ for some $E_j \subseteq A_j$, hence $V(E) \cap X_i$ is a set of all prime ideals in X_i containing E_i, which implies $\phi_i^*(V(E) \cap X_i) = \{\mathfrak{p} \in V(E_i)\} = V(E_i)$. Thus, ϕ^* is closed map. Moreover, ϕ_i^* is bijection, since $\phi_i^*(A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n) = \mathfrak{p}$ which gives an injection, (two distinct elements in X_i has two distinct prime ideal part of i-th position, thus their image is different.) Also every prime ideal in A_i is also image of ϕ^* by construction. Hence ϕ^* is bijectively continuous and closed map. Hence it is homeormorphism. (Notes that $\pi_i : A \to A_i$ gives the inverse map; since $\pi^{-1}(\mathfrak{p}) = A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n$.)

Now we claim that $\mathrm{Spec}(A) = \amalg_{i=1}^n X_i$. Notes that actually as a subset, $X_i \cap X_j = \emptyset$ if $i \neq j$, since A_i and A_j are not prime. Thus it suffices to show that $\mathrm{Spec}(A) = \bigcup_{i=1}^n X_i$ and to check its disjoint topology structure, i.e., subset U of $\mathrm{Spec}(A)$ is open if and only if $\pi_i^{*-1}(U)$ is open in $\mathrm{Spec}(A_i)$, where $\pi_i : A \to A_i$ be canonical projection. This statement is equivalent to saying that subset F of $\mathrm{Spec}(A)$ is closed if and only if $\pi_i^{*-1}(F)$ is closed in $\mathrm{Spec}(A_i)$. (To see this, notes that LHS is equivalent to say that U^c is closed, and RHS is equivalent to say that $\pi_i^{*-1}(U^c)$ is closed, from the injectivity of the map π_i^*.)

To see that every prime ideal in $\mathrm{Spec}(A)$ has a form $A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n$, let P be a prime ideal in A. Let $e_1, \cdots, e_n$ be an element such that e_k is k-th position is 1 and the other positions are zeros. Then, if P has all e_i, the $P = A$, not a prime. So P doesn't have some e_i. Fix the smallest e_i which is not in P. Then, for any $j \neq i$, $e_ie_j = 0 \in P$ implies $e_j \in P$.

Hence, $P = A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n$ where $\mathfrak{p}$ is projection of P onto A_i. To see that $\mathfrak{p}$ is prime ideal in A_i, notes that

$$A/P \cong A_1/A_1 \times \cdots \times A_{i-1}/A_{i-1} \times A_i/\mathfrak{p} \times A_{i+1}/A_{i+1} \times \cdots \times A_n/A_n \cong A_i/\mathfrak{p}$$

is integral domain. Thus $\mathfrak{p}$ should be prime ideal.

To see that $\mathrm{Spec}(A)$ has a disjoint union topology, let F be a closed set in $\mathrm{Spec}(A)$. Then $F = V(E)$ for some subset $E = \prod_{i=1}^{n} E_i$ of A. Then

$$\pi_i^{*-1}(V(E)) = \pi_i^{*-1}(V(E) \cap X_i) = \{\pi_i(x) : x \in V(E) \cap X_i\} = \{\mathfrak{p} \in \mathrm{Spec}(A_i) : E_i \subseteq \mathfrak{p}\} = V(E_i).$$

Conversely, let F be a set such that $\pi_i^{*-1}(F)$ is closed for all i. Then, $\pi_i^{*-1}(F) = V(E_i)$ for some subset E_i of A_i. We claim that $F = V(E)$. To see this, π_i^* is actually homeomorphism between $\mathrm{Spec}(A_i)$ and X_i, thus

$$\pi_i^*(\pi_i^{*-1}(F)) = F \cap X_i$$

and $F \cap X_i = \{A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n : E_1 \subset \mathfrak{p} \in \mathrm{Spec}(A_i)\} \subseteq V(E)$. Hence, $F = \bigcup_{i=1}^{n} F \cap X_i \subseteq V(E)$. Conversely, if $P = A_1 \times \cdots \times A_{i-1} \times \mathfrak{p} \times A_{i+1} \times \cdots \times A_n \in V(E)$, then $\mathfrak{p}$ contains E_i, thus $P \in X_i \cap F$. This implies $F = V(E)$, done.

(b) (Second problem) If $X = \mathrm{Spec}(A)$ is disconnected, then there is two disjoint clopen sets whose union is $X = \mathrm{Spec}(A)$. Thus $X = V(I) \cup V(J)$ for some ideal I and J with $V(I) \cap V(J) = \emptyset$. By exercise 15 iii), $V(I + J) = \emptyset$. Hence $I + J$ does not lie in any prime ideal, thus it cannot be a proper ideal; otherwise it has a maximal ideal containing itself. Thus $I + J = A$. Also, $X = V(I) \cup V(J) = V(IJ)$. This implies IJ is contained in any prime ideal, hence $IJ \subseteq \mathfrak{R}$. Also, from the coprime condition of I and J, $IJ = I \cap J$ (see [3][p.7].)

Now we want to see *iii*). Let $a \in I, b \in J$ such that $a + b = 1$. (Notes that a, b are nonzero and not 1.) Then, ab is nilpotent, thus $\exists n \in \mathbb{N}$ such that $a^n b^n = 0$. Now think about $(a + b)^{2n}$. Then, we can let

$$e_1 = a^{2n} + \binom{2n}{1} a^{2n-1} b + \cdots + \binom{2n}{n-1} a^{n+1} b^{n-1}, e_2 = \binom{2n}{n+1} a^{n-1} b^{n+1} + \cdots + \cdots \binom{2n}{2n-1} ab^{2n-1} + b^{2n}.$$

Then, $e_1 + e_2 + \binom{2n}{n} a^n b^n = e_1 + e_2 = 1$ and $e_1 e_2 = 0$ since every term in $e_1 e_2$ contains $a^n b^n$. Hence, $e_1(1 - e_1) = 0$ implies $e_1(e_1 - 1) = 0$ thus $e_1^2 = e_1$. Similarly, $e_2^2 = e_2$. Also we know that $e_1 \in I, e_2 \in J$ thus they are nontrivial idempotent.

Next, we assume iii) and prove ii). Suppose t be an idempotent. Then, $s = 1 - t$ is also idempotent since $s^2 = t^2 - 2t - 1 = 1 - t$. Hence, let S, T be a subring of A generated by s and t, i.e., $S = sA$ and $T = tA$. Let $\phi : A \to S \times T$ by $x \mapsto (sx, tx)$. It is surjective, since for any (sx, ty), take $h = sx + ty$. Then, $sh = s^2 x + sty = sx, th = ty$, done. Also, it is injective, since $(sx, tx) = 0$ implies that $tx = 0, (1 - t)x = 0$. Hence, $x = tx = 0$. Thus it is bijective homomophism, so isomorphism.

From ii) to i) is just done by First problem part.

(c) In particular, the spectrum of a local ring is connected since Exercise 12 shows that it has no nontrivial idempotent.

23. (a) X_f is open by Exercise 17. So it suffices to show that $X_f = V(f)^c$ is closed. It is equivalent to say that $V(f)$ is open. To see this, let $f \in A$. Then $s = 1 - f$ is also idempotent. We claim that $V(f) \cap V(s) = \emptyset$. If $\mathfrak{p} \in V(f) \cap V(s)$, then $\mathfrak{p}$ contains f and $s = 1 - f$, thus $\mathfrak{p}$ contains an ideal generated by f and s, which implies $f + s = 1 \in \mathfrak{p}$, contradiction. Also, $V(f) \cup V(s) = V(fs) = V(0) = \mathrm{Spec}(A)$ by exercise 15. Hence, $V(s)$ and $V(f)$ are disjoint closed sets whose union is $\mathrm{Spec}(A)$. Thus $V(s) = V(f)^c, V(f) = V(s)^c$. This implies that $V(f), V(s)$ are clopen. This implies X_f and X_s are clopen, too. Since f was arbitrarily chosen, every basic open sets are clopen.

(b) Notes that

$$X_{f_1} \cup \cdots \cup X_{f_n} = \bigcup_{i=1}^{n} V(f_i)^c = \left(\bigcap_{i=1}^{n} V(f_i) \right)^c = (V(\{f_i\}_{i=1}^n))^c \,,$$

where last equality comes from Exercise 15 iii). By exercise 11 iii), an ideal generated by $\{f_i\}_{i=1}^n$ is principal, so $(f) = (\{f_i\}_{i=1}^n)$ for some $f \in A$. Hence,

$$X_{f_1} \cup \cdots \cup X_{f_n} = (V(\{f_i\}_{i=1}^n))^c = V(f)^c = X_f.$$

(c) Let $Y \subseteq \mathrm{Spec}(A) = X$ be clopen. Since Y is open, Y is union of basic open sets. Since Y is closed and X is quasi-compact by Exercise 17 v), Y is quasi-compact. Hence Y is a finite union of basic open sets by Exercise 17 vii). By above ii), done.

(d) Notes that $\mathrm{Spec}(A)$ is quasi-compact by Exercise 17 v). Now take $x, y \in \mathrm{Spec}(A)$ which are distinct. Then without loss of generality, assume $\mathfrak{p}_x \not\subseteq \mathfrak{p}_y$. Then $z \in \mathfrak{p}_x \setminus \mathfrak{p}_y$. Then, $V(z)$ contains $\mathfrak{p}_x$, $V(z)^c$ contains $\mathfrak{p}_y$. Since they are open, so $\mathrm{Spec}(A)$ is Hausdorff.

24. Notes that join and meets in the lattice are associative. Hence, addition and multiplication are well-defined. Also, $a^2 = a \wedge a = a$, done.

    Conversely, let A be a given boolean ring. Then, $a(a+b+ab) = a + 2ab = a$ implies $a \leq a+b+ab$. Similarly, $b \leq a+b+ab$. Thus, $a \wedge b \leq a+b+ab$. Moreover, if $a \leq c$ and $b \leq c$, then $(a+b+ab)c = ac + bc + abc = a + b + ab$ implies that $a + b + ab \leq c$. Hence, $a \wedge b = a + b + ab$. Thus $a \wedge b$ exists. Conversely, $aba = ab$ and $abb = ab$ implies $ab \geq a, b$. Also, for any $c \geq a, b$, this implies $ac = a, bc = b$, thus $abc = ab$, which implies $c \geq ab$. This shows that $ab = a \vee b$. Now, $a \wedge a' = a + (1-a) + a(1-a) = 1$, $a \vee a' = a(1-a) = 0$, done.

25. Let B be a Boolean lattice. Then using Execise 24, we can regard it as a Boolean ring, say A. Now take a map $A \to \mathrm{Spec}(A)$ by $f \mapsto V(f)$. Then, $f \leq g$ implies $f = fg$ implies $V(f) \supseteq V(g)$ since $f = fg$ implies that all $\mathfrak{p} \in V(g)$ contains $fg = f$, which implies $\mathfrak{p} \in V(f)$. Hence $X_f \subseteq X_g$, which implies $X_f \leq X_g$ if we regard topology as a lattice ordered by inclusion. Also, complement, sup and inf are preserved, hence lattice of topology of $\mathrm{Spec}(A)$ is isomorphic to its Boolean lattice as a lattice. Since the topology is open and closed subsets of compact Hausdorff topology by Exercise 23 iv), done.

26. This problem shows that if X is compact Hausdorff and $C(X)$ is the ring of all real-valued continuous function on X, then $Max(C(x)) = \{\mathfrak{m}_x : x \in X\}$ where $\mathfrak{m}_x$ is a set of all functions in $C(X)$ vanishing at x. In the step iii), we claim that $\mu(U_f) = \tilde{U}_f$. Notes that $\mu(U_f) = \{\mathfrak{m}_x : x \in U_f\}$. Thus if $\mathfrak{m}_x \in \mu(U_f)$, then $f \notin \tilde{m}_x$ since $f(x) \neq 0$, thus $\mathfrak{m}_x \in \tilde{U}_f$. Conversely, if $\mathfrak{m} \in \tilde{U}_f$, then by bijectivity of μ, $\mathfrak{m} = \mathfrak{m}_x$ for some $x \in X$. Hence $f(x) \neq 0$. This implies $\mathfrak{m} \in \mu(U_f)$.

27. We need the weak Nullstellensatz, stating that

    Claim V. *V If k is algebraically closed field, and an ideal $\mathfrak{a}$ of $k[t_1, \cdots, t_n]$ is not (1), then $Z(\mathfrak{a}) \neq \emptyset$.*

    This is result of Exercise 5.17. Assume this and let $\mathfrak{m}$ be the maximal ideal of $k[t_1, \cdots, t_n]$. Then, $\mathbb{Z}(\mathfrak{m})$ is nonzero by the weak Nullstellensatz. Hence, $\exists x \in \mathbb{Z}(\mathfrak{m})$. Thus $\mathfrak{m}_x$ contains $\mathfrak{m}$. By the maximality of $\mathfrak{m}$, $\mathfrak{m} = \mathfrak{m}_x$.

    Now for $P(X) = k[t_1, \cdots, t_n]/I(X)$, we know that $\mathrm{Spec}(P(X))$ has 1-1 order preserving correspondence with ideals containing $I(X)$. Hence if $\overline{\mathfrak{m}}$ is the maximal ideal in $P(X)$, then it is just projection of maximal ideal in $k[t_1, \cdots, t_n]$ containing $I(X)$. By above result, we can assume this maximal ideal is $\mathfrak{m}_x$ for some $x \in k^n$. Since $I(X) \subseteq \mathfrak{m}_x$, so $x \in X$. Hence, any maximal ideal in $P(X)$ is of the form $\mathfrak{m}_x$ for $x \in X$.

    Also notes that $\mathfrak{m}_x = (\xi_i - x_i)_{i=1}^n$, since the RHS is subideal of $\mathfrak{m}_x$ and maximal from the fact that $P(X)/(\xi_i - x_i)_{i=1}^n \cong k$ by a map sending $\xi_i \mapsto x_i$.

28. Let $\phi^{\#} : P(Y) \to P(X)$ be a map induce by ϕ. First of all, it is k-algebra homomorphism; to see this, for any $a \in k$, $b, c \in P(Y)$,

$$\begin{aligned} \phi^{\#}(ab) &= ab \circ \phi = a(b \circ \phi) = a\phi^{\#}(b) \\ \phi^{\#}(b+c) &= (b+c) \circ \phi = (b \circ \phi) + (c \circ \phi) = \phi^{\#}(b) + \phi^{\#}(c) \\ \phi^{\#}(bc) &= (bc) \circ \phi = (b \circ \phi)(c \circ \phi) = \phi^{\#}(b)\phi^{\#}(c) \end{aligned}$$

where last equation comes from the fact that $bc(y) = b(y)c(y)$ for any $y \in Y$.

Now let $\mathrm{Hom}(X, Y)$ be a set of all regular maps from X to Y and $\mathrm{Hom}_k(P(Y), P(X))$ be set of all k-algebra homomorphisms from $P(Y)$ to $P(X)$. As we saw above, $(\cdot)^{\#}$ is well-defined. To see it is injective, let $\phi^* = \psi^*$. If we let coordinates of $\phi(x)$ and $\psi(x)$ as $\phi_i : k^n \to k$ and $\psi_i : k^n \to k$ respectively for $i = 1, \cdots, m$, then for any coordinate functions $\xi_i \in P(Y)$,

$$\phi_i = \xi_i \circ \phi = \phi^{\#}(\xi_i) = \psi^{\#}(\xi_i) = \xi_i \circ \psi = \psi_i$$

for all $i \in [m]$. This implies $\phi = \psi$ as a polynomial mapping.

Moreover, $(\cdot)^{\#}$ is surjective. let $\lambda \in \mathrm{Hom}_k(P(Y), P(X))$. Let $\pi : k[t_1, \cdots, t_m] \to P(Y)$ be canonical projection. Then $\lambda \circ \pi : k[t_1, \cdots, t_m] \to P(X)$. Hence Let $\phi_i := \lambda \circ \pi(t_i) \in P(X)$ for all $i \in [m]$. Then $\phi_i : X \to k$ is regular function since it has preimage in $k[t_1, \cdots, t_n]$ from the fact that $P(X)$ is the quotient ring of $k[t_1, \cdots, t_n]$.

Then, we know that for any $t_i \in k[t_1, \cdots, t_m]$, $\lambda \circ \pi(t_i) = \phi_i$. Thus for any $\eta(t_1, \cdots, t_m) \in k[t_1, \cdots, t_m]$,

$$\lambda \circ \pi(\eta) = \eta(\lambda \circ \pi(t_1), \cdots, \lambda \circ \pi(t_m)) = \eta(\phi_1, \cdots, \phi_m) = \eta \circ \phi.$$

since each π and λ are multiplicative homomorphism (in different sense; λ as k-algebra and π as ring homomorphism.) Now it suffices to show that for any $\eta' \in \eta + I(X)$, $\eta \circ \phi = \eta' \circ \phi$. If we show this, then $\phi^{\#}$ is well-defined on $\pi(\eta)$, and λ and ϕ coincides on every $\pi(\eta) \in P(Y)$, so $\lambda = \phi$. And the statement for any $\eta' \in \eta + I(Y)$, $\eta \circ \phi = \eta' \circ \phi$ is equivalent to say that $\psi \circ \phi = 0$ for any $\psi \in I(Y)$. To see this, notes that

$$\psi \circ \phi = \psi(\phi_1, \cdots, \phi_m) = \psi(\lambda \circ \pi(t_1), \cdots, \lambda \circ \pi(t_m)) = \lambda \circ \pi(\psi) = \lambda(0) = 0$$

since $\psi \in I(Y)$.

Thus $(\cdot)^{\#}$ is surjective. Actually, for $\phi : X \to Y$ and $\psi : Y \to Z$, and for any $\eta \in P(Z)$,

$$(\psi \circ \phi)^{\#}(\eta) = \eta \circ \psi \circ \phi = \phi^{\#}(\eta \circ \psi) = \phi^{\#}(\psi^{\#}(\eta)) = \phi^{\#} \circ \psi^{\#}(\eta).$$

Hence, this shows that the coordinate ring is contravariant functor from the category of affine algebraic varieties and regular maps to the category of finitely generated k-algebras and k-algebra homomorphism.

2 Modules

Exercise 2.2. *1.* $\mathrm{Ann}(M + N) = \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$.

2. $(N : P) = \mathrm{Ann}((N + P)/N)$.

Proof. If $x \in \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$, then $x(M + N) = 0$, since $M + N$ is just finite sum on $M \cup N$. So $x \in \mathrm{Ann}(M + N)$. Conversely, $x \in \mathrm{Ann}(M + N)$ implies that $xm = 0$ and $xn = 0$ for all $m \in M, n \in N$ since $M, N \subseteq M + N$.

For the second one, let $x \in (N : P)$. For an arbitrary element $z \in (N + P)/N$, z has representation $z = \sum_{i=1}^m n_i + \sum_{k=1}^l p_i + N$ for some $n_i \in N, p_i \in N$, hence $xz = \sum_{i=1}^m xn_i + \sum_{k=1}^l xp_i + xN = 0 + N = 0$.Thus $x \in \mathrm{Ann}((N + P)/N)$. Conversely, if $x \in \mathrm{Ann}((N + P)/N)$, then $xp \in N$ for any $p \in P$, otherwise $p + N$ is not annihilted by x, contradiction. Thus $x \in (N : P)$. $\square$

Notes that every ideal is subring, but ideal as a subring may not be commutative ring with unity. For example, think an ideal of direct product generated by elements $(0, \cdots, 0, a, 0, \cdots, 0)$. This ideal is definitely a subring of A, but it may not have unity.

Corollary 2.7. *Let M be a finitely generated A-module, N a submodule of M, $\mathfrak{a} \subseteq \mathfrak{R}$ is an ideal contained in the Jacobson radical. Then, $M = \mathfrak{a}M + N \implies M = N$.*

Proof. Notes that $\mathfrak{a}(M/N) = (\mathfrak{a}M + N)/N$ as a submodule of M/N. To see this, let $m + N \in \mathfrak{a}(M/N)$. Then, $m = am'$ for some $a \in \mathfrak{a}$ and $m' \in M$, thus $m + N = am' + 0 + N$ for $0 \in N$, hence $m + N \in (\mathfrak{a}M + N)/N$. Conversely, let $am + n + N$ be an element in $(\mathfrak{a}M + N)/N$. Then, $am + n + N = am + N$, thus $am + n + N \in \mathfrak{a}(M/N)$.

Thus the given condition says that $\mathfrak{a}(M/N) = (\mathfrak{a}M + N)/N = M/N$. Hence by Corollay 2.6, $M/N = 0$. This implies $M = N$. ☐

Proposition 2.9. *1. Let*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$

be a sequence of A-modules and homomorphisms. Then, the sequence is exact if and only if for all A-module N, the sequence

$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{\overline{v}} \mathrm{Hom}(M, N) \xrightarrow{\overline{u}} \mathrm{Hom}(M', N)$$

is exact.

2. Let

$$0 \to N' \xrightarrow{u} N \xrightarrow{v} N''$$

be a sequence of A-modules and homomorphisms. Then, the sequence is exact if and only if for all A-module M, the sequence

$$0 \to \mathrm{Hom}(M, N') \xrightarrow{\overline{u}} \mathrm{Hom}(M, N) \xrightarrow{\overline{v}} \mathrm{Hom}(M, N'')$$

is exact.

Proof. Suppose that

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$

is exact. Then, u is injective and v is surjective. Thus $\overline{v}$ is injective since if $f, g \in \mathrm{Hom}(M'', N)$ such that $f \circ v = g \circ v$, then for any $x \in M''$, $\exists x' \in M$ such that $v(x') = x$, hence $f(x) = f \circ v(x') = g \circ v(x') = g(x)$. Also, from exactness of sequence of modules, $\mathrm{Im}(\overline{v}) \subseteq \ker(\overline{u})$ since for any $f \in \mathrm{Hom}(M'', N)$, $f \circ v \circ u = f \circ 0 = 0$. Now let $f \in \ker(\overline{u})$. Then, $\ker f \supseteq \mathrm{Im}\, u = \ker v$. Hence, by argument in [3][p.19], f give rise to $\bar{f} : M/\ker(v) \to N$. Since $M/\ker(v) \cong M''$, there exists $g : M'' \to N$ whose behavior is equal to $\bar{f}$. Now notes that $f(x) = \bar{f}(x + \ker(v)) = g \circ v(x)$, hence f is in $\ker(\overline{v})$, done.

Conversely, let

$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{\overline{v}} \mathrm{Hom}(M, N) \xrightarrow{\overline{u}} \mathrm{Hom}(M', N)$$

be exact for any N. Then, to see v is surjective, suppose that v is not surjective. This implies $\mathrm{Im}(v)$ is proper submodule of M''. Then, take $N = M''/\mathrm{Im}(v)$. Since N is nonzero module, $\pi : M \to M''/\mathrm{Im}(v)$, which is canonical surjection, is nonzero. Hence, $\overline{v}(\pi) \neq 0$. However, $\pi \circ v = 0$ since $\mathrm{Im}(v) \subseteq \ker(\pi)$, contradiction. So v is surjective. Moreover, $\overline{u} \circ \overline{v} = 0$ implies $f \circ v \circ u = 0$ for any $f \in \mathrm{Hom}(M'', N)$. Take $N = M''$ and f be an identity map. Then, $f \circ v \circ u = 0$ implies $v \circ u = 0$, since f is isomorphism. This implies $\mathrm{Im}(u) \subseteq \ker(v)$. To see equality, let $N = M/\mathrm{Im}(u)$ an let $\phi : M \to N$ be the projection. Then $\phi \in \ker(\overline{u})$, thus $\exists \psi : M'' \to N$ such that $\psi = \phi \circ v$ Thus $\mathrm{Im}(u) = \ker(\phi)$ and $\ker(\phi)$ contains $\ker(v)$. This implies $\mathrm{Im}(u) = \ker(v)$.

For the second one, suppose

$$0 \to N' \xrightarrow{u} N \xrightarrow{v} N''$$

be exact. Then, $\mathrm{Im}(u) = \ker(v)$ and u is injective. To see $\overline{u}$ is injective, let $f \in \mathrm{Hom}(M, N')$ such that $\overline{u}(f) = 0$. Then $u \circ f = 0$. This implies $f = 0$ since u is injective. Also, $\mathrm{Im}(\overline{u}) \subseteq \ker(\overline{v})$ from $v \circ u = 0$. If $f \in \ker(\overline{v})$, then $v \circ f = 0$. Thus $\mathrm{Im}(f) \subseteq \ker(v) = \mathrm{Im}(u)$. Hence, define

$$\bar{f} = u^{-1} \circ f.$$

It is well-defined since $\mathrm{Im}(f) \subseteq \mathrm{Im}(u)$ and u is injective. Also, $\bar{f} \in \mathrm{Hom}(M, N')$ and $\overline{u}(\bar{f}) = u \circ u^{-1} \circ f = f$. Hence $f \in \mathrm{Im}(u)$, done.

Conversely, suppose

$$0 \to \mathrm{Hom}(M, N') \xrightarrow{\overline{u}} \mathrm{Hom}(M, N) \xrightarrow{\overline{v}} \mathrm{Hom}(M, N'')$$

be exact for any M. Then, $\overline{u}$ is injective and $\mathrm{Im}(\overline{u}) = \ker(\overline{v})$. To see u is injective, let $M = \mathbb{Z}$ and $x \in \ker(u)$. Then we have a map $f : \mathbb{Z} \to N'$ by $1 \mapsto x$. Then $u \circ f = 0$ implies $f \in \ker(\overline{u})$ thus $f = 0$. Thus $x = 0$. Hence $\ker(u) = 0$, so u is injective. Moreover, $\mathrm{Im}(u) \subseteq \ker(v)$ since $\overline{v} \circ \overline{u} = 0$ implies $v \circ u \circ f = 0$ for any $f : M \to N'$, therefore by taking $M = N'$ and f be identitiy, we get $v \circ u \circ f = 0$ implies $v \circ u = 0$. Also, to see $\mathrm{Im}(u) \supseteq \ker(v)$, let $x \in \ker(v)$. Then let $M = \mathbb{Z}$ and take a map $f : \mathbb{Z} \to N$ by $1 \mapsto x$. Then,

$$\mathrm{Im}(\overline{v}(f)) = \mathrm{Im}(v \circ f) = 0$$

implies that $f \in \ker(\overline{v}) = \mathrm{Im}(\overline{u})$. Hence there exists $g \in \mathrm{Hom}(\mathbb{Z}, N')$ such that $u \circ g = f$. Thus, $u(g(1)) = f(1) = x$. This implies $x \in \mathrm{Im}(u)$, done. □

Proposition 2.14. *Let M, N, P be A-module. Then there exists unique isomorphisms*

1. $M \bigotimes N \to N \bigotimes M$
2. $(M \bigotimes N) \bigotimes P \to M \bigotimes (N \bigotimes P) \to M \bigotimes N \bigotimes P$
3. $(M \bigoplus N) \bigotimes P \to (M \bigotimes P) \bigoplus (N \bigotimes P)$
4. $A \bigotimes M \to M$

such that, respectively,

1. $x \otimes y \mapsto y \otimes x$
2. $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z) \mapsto x \otimes y \otimes z$
3. $(x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$
4. $a \otimes x \mapsto ax$.

Proof. Second one was prove in the book [3]. For the first one, $M \times N \to N \times M \to N \bigotimes M$ gives a bilinear map sending (x, y) to $y \otimes x$, so by the universal property of tensor product it induces a map $M \bigotimes N \to N \bigotimes M$ sending $x \otimes y$ to $y \otimes x$. Similarly, $N \times M \to M \times N \to M \bigotimes N$ is a bilinear map sending (y, x) to $x \otimes y$, so by the universal property, we hvae a map $y \otimes x$ to $x \otimes y$. Those two maps are inverse of each other, done.

For the third one, $M \bigoplus N \times P \to M \times P \oplus N \times P \to M \bigotimes P \oplus N \bigotimes P$ is a bilinear map, so we have a linear map $\phi : M \bigoplus N \bigotimes P \to M \bigotimes P \oplus N \bigotimes P$ by $(m, n) \otimes p \mapsto (m \otimes p, n \otimes p)$. On the other hand, $M \times P \to (M \oplus N) \bigotimes P$ and $N \times P \to (M \oplus N) \bigotimes P$ by $(m, p) \mapsto (m, 0) \otimes p$ and $(n, p) \mapsto (0, n) \otimes p$ are bilinear map, so we have two linear maps $M \bigotimes P \to (M \oplus N) \bigotimes P$ and $N \bigotimes P \to (M \oplus N) \bigotimes P$. Thus its direct product has a map

$$\psi : M \bigotimes P \oplus N \bigotimes P \to M \bigoplus N \bigotimes P \text{ by } (m \otimes p, n \otimes p') = (m, 0) \otimes p + (0, n) \otimes p'.$$

Notes that its restriction on each side is linear, hence this map itself is bilinear. Now notes that

$$\phi \circ \psi(m \times p, n \oplus p') = \phi((m, 0) \otimes p + (0, n) \otimes p') = \phi((m, 0) \otimes p) + \phi((0, n) \otimes p') = (m \otimes p, 0 \otimes p) + (0 \otimes p', n \otimes p') = (m \otimes p, n \otimes p')$$

and

$$\psi \circ \phi((m, n) \otimes p) = \psi(m \otimes p, n \otimes p) = (m, 0) \otimes p + (0, n) \otimes p = (m, n) \otimes p.$$

Hence they are inverse of each other, thus isomorphism.

For the last one, notes that $A \times M \to M$ by $(a, m) \to am$ is bilinear. Hence it induces a bilinear map $A \bigotimes M \to M$ by $a \otimes m \mapsto am$. Hence this map induces a linear map $\phi : A \bigotimes M \to M$. by $a \otimes m \to am$. Now think about the map $\psi : M \to A \bigotimes M$ by $m \mapsto 1 \otimes m$. Then, $\phi \circ \psi(m) = \phi(1 \otimes m) = m$ and $\psi \circ \phi(a \otimes m) = \psi(am) = 1 \otimes am = a \otimes m$, done. □

Exercise 2.15. *Let A, B be rings, let M be an A-module, P a B-module, N an (A, B)-bimodule (that is, N is simultaneously an A-module and a B-module and the two structures are compatible in the sense that $a(xb) = (ax)b$ for all $a \in A, b \in B, x \in N$). Then $M \bigotimes_A N$ is naturally a B-module, $N \bigotimes_B P$ an A-module, and we have*

$$(M \bigotimes_A) \bigotimes_B P \cong M \bigotimes_A (N \bigotimes_B P).$$

Proof. Notes that for given $p \in P$, $M \times N \to M \bigotimes_A (N \bigotimes_B P)$ by $(m, n) \mapsto m \otimes (n \otimes p)$ is bilinear. Hence, it induces a linear map $M \bigotimes_A N \to M \bigotimes_A (N \bigotimes_B P)$ by $m \otimes n \mapsto m \otimes (n \otimes p)$ for given p. Also, we can construct a amp

$$(M \bigotimes_A N) \times P \to M \bigotimes_A (N \bigotimes_B P) \text{ by } (m \otimes n, p) \mapsto m \otimes (n \otimes p).$$

This is B-linear with respect to fixed $(m \otimes n)$ (check.) And also A-linearity on fixed p is just showed. Hence we can get a (A, B)-linear map

$$(M \bigotimes_A N) \bigotimes_B P \to M \bigotimes_A (N \bigotimes_B P) \text{ by } (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p).$$

We can do the symmetric argument by interchanging M, N, and P to get an isomorphism as (A, B)-bimodule. □

Exercise 2.20. *If $f : A \to B$ is a ing homomorphism and M is a flat A-module, then $M_B = B \bigotimes_A M$ is a flat B-module.*

Proof. Let $j : N \to N'$ be any injective B-module homomorphism. It suffices to show that $1 \otimes j : M_B \bigotimes_B N \to M_B \bigotimes_B N'$ is also injective. Since $f : A \to B$ exists, we can think N, N' are A-modules. Then,

$$j \otimes 1_M : N \bigotimes_A M \to N' \bigotimes_A M$$

is injective as A-module map since M is a flat A-module. Then,

$$N \bigotimes_B M_B \underbrace{=}_{\text{Def.}} N \bigotimes_B (B \bigotimes_A M) \underbrace{\cong}_{\text{by 2.15 as bimodule}} (N \bigotimes_B B) \bigotimes_A M \underbrace{\cong}_{\text{by 2.14 as bimodule}} N \bigotimes_A M.$$

Notes that last isomorphism as bimodule comes from the fact that for given $f : A \to B$ a ring homomorphism and $W \cong Q$ as a B-module, then they are isomorphism as an A-module since the given isomorphism map is also can be regarded as A-module homomorphism satisfying surjectivity and injectivity.

Thusm the map $j \otimes 1 : N \bigotimes_B M_B \to N' \bigotimes_B M_B$ is injective if and only if $\phi : N \bigotimes_A M \to N' \bigotimes_A M$ is injective as a B-module. We already know that as an A-module homomorphism ϕ is injective by the fact that M is flat. Since injectivity do not depends on its module structure, this map is also injective as a B-module, since this map already has a bimodule homomorphism as we've shown in the equation. □

Claim VI. *p.30 Let $f : A \to B, g : A \to C$ be two ring homomorphisms. Let $h : B \to C$ be a ring homomorphism. Then, h is A-algebra map if and only if $h \circ f = g$.*

Proof. Suppose $h \circ f = g$. Let $x, y \in B$. Then h is additive since it is ring homomorphism. For any $a \in A, b \in B$,

$$a.h(b) = g(a)h(b) = h(f(a))h(b) = h(f(a)b) = h(a.b)$$

Thus h is A-module map. Conversely, suppose h is A-algebra homomorphism. Then, for any $a \in A$,

$$h(f(a)) = h(a.1) = a.h(1) = g(a)h(1) = g(a).$$

Thus $h \circ f = g$. $\square$

Claim VII. *p.31 In the last paragraph, the diagram commutes when we set $u(b) = b \otimes 1$ and $v(c) = 1 \otimes c$.*

Proof. $u \circ f(a) = f(a) \otimes 1 = a.(1 \otimes 1) = 1 \otimes g(a) = v \circ g(a)$. $\square$

1. By Bezout's theorem, there exists $a, b \in \mathbb{Z}$ such that $am + bn = 1$. Thus for $x \otimes y \in (\mathbb{Z}/m\mathbb{Z}) \bigotimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$,

$$\overline{x} \otimes \overline{y} = (am + bn)(\overline{x} \otimes \overline{y}) = \overline{amx} \otimes y + x \otimes \overline{bmy} = 0 \otimes y + x \otimes 0 = 0.$$

2. Notes that $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ is exact as a sequence of A-modules. By tensoring with M, we can get

$$0 \to \mathfrak{a} \bigotimes M \to A \bigotimes M \to A/\mathfrak{a} \bigotimes M \to 0.$$

   By Proposition 2.18, $\mathfrak{a} \bigotimes M \xrightarrow{j} A \bigotimes M \xrightarrow{i} A/\mathfrak{a} \bigotimes M \to 0$ is exact. Thus, i is surjective, hence

$$A/\mathfrak{a} \bigotimes M \cong A \bigotimes M/\ker(i) = A \bigotimes M/\operatorname{Im}(j) \cong M/\operatorname{Im}(j).$$

   Notes that $\operatorname{Im}(j) = \mathfrak{a}M$, done.

3. Let $M_k = k \bigotimes_A M \cong M/\mathfrak{m}M$ by Exercise 2. Then $M_k = 0$ implies $M = kM$, and by Nakayama lemma this implies $M = 0$. However,

$$\begin{aligned}
M \bigotimes_A N = 0 &\implies (M \bigotimes_A N)_k = 0 \\
&\implies k \bigotimes_A (M \bigotimes_A N) = 0 \\
&\implies (k \bigotimes_A k) \bigotimes_A (M \bigotimes_A N) = 0 \\
&\implies k \bigotimes_A M_k \bigotimes_A N \text{ by Proposition 2.14 i) and definition of } M_k \\
&\implies M_k \bigotimes_A k \bigotimes_A N \text{ by Proposition 2.14 i)} \\
&\implies M_k \bigotimes_A N_k = 0.
\end{aligned}$$

   Notes that $M_k \bigotimes_A \cong N_k \cong M_k \bigotimes_k \cong N_k$ as A-module since for any two representative a and b of the same equivalence class in $k = A/\mathfrak{m}$, $a = m+b$ for some $m \in \mathfrak{m}$, thus from $M_k \bigotimes_A N_k = k \bigotimes_A M \bigotimes_A k \bigotimes_A N$, if we take $k_1 \otimes m \otimes k_2 \otimes n \in M_k \bigotimes_A N_k$,

$$a(k_1 \otimes m \otimes k_2 \otimes n) = \overline{a}k_1 \otimes m \otimes k_2 \otimes n = \overline{b}k_1 \otimes m \otimes k_2 \otimes n = b(k_1 \otimes m \otimes k_2 \otimes n).$$

Thus, acting by $a \in A$ is equivalent to acting by $\bar{a} \in k$ which is an equivalence class containing a. Thus, $M_k \bigotimes_A N_k \cong M_k \bigotimes_k N_k$ as an A-module. Thus given condition implies that

$$M_k \bigotimes_k N_k = 0$$

However, $M_k \cong M/\mathfrak{m}M$ and $N_k \cong N/\mathfrak{m}N$ are naturally k-module, i.e., k-vector space by [3][p.22]. Thus, they are vector space. And $M_k \bigotimes_k N_k$ is still k-module, thus it is also a vector space. Its dimension is $\dim M_k \times \dim N_k$ as a vector space (See Do Carmo to prove this fact by constructing basis. Proving that tensors of bases form a basis needs a fact of dual basis.) Hence either $M_k = 0$ or $N_k = 0$. Then by Nakayama's lemma, either $M = 0$ or $N = 0$.

4. We need a claim for direct sum.

Claim VIII. *Let $M = \bigoplus_{i \in I} M_i$, $N = \bigoplus_{i \in I} N_i$ be two direct sums with the same index. Let $f_i : M_i \to N_i$ be a module map. Then, there exists $f : M \to N$ a module map such that*

$$\pi_{N,j} \circ f \circ \iota_{M,i} = \begin{cases} f_i & \text{if } i = j \\ 0 & o.w. \end{cases}$$

Moreover f is injective (or surjective) if and only if all f_i's are injective (or surjective, resp.)

Proof. From the universal property of direct sum, from $\iota_{N,i} \circ f_i : M_i \to N$, we can have a map $f : M \to N$ such that $f \circ \iota_{M,i} = \iota_{N,i} \circ f_i$. Hence, By applying $\pi_{N,i}$ for each side, we get $\pi_{N,j} \circ f \circ \iota_{M,i} = \pi_{N,i} \circ \iota_{N,i} \circ f_i = f_i$. If we apply $\pi_{N,j}$ with $j \neq i$, then $\pi_{N,j} \circ \iota_{N,i} = 0$ implies the second case.

Moreover, f is injective if and only if all f_i is injective from the equation $f \circ \iota_{M,i} = \iota_{N,i} \circ f_i$. (Notes that composition is injective if and only if its rightmost part is injective. Also, from

$$f \circ \iota_{M,i} = f_i \implies f \circ \iota_{M,i} \circ \pi_{M,i} = f_i \circ \pi_{M,i} \implies f = f_i \circ \pi_{M,i},$$

f is surjective if and only if f_i is surjective, since composition is surjective if and only if its leftmost part is surjective. ☐

Let $j : N \to N'$ be injective A-module map. Let $j \otimes 1 : N \bigotimes_A M \to N' \bigotimes_A M$. Notes that $N \bigotimes_A M \cong \bigoplus_{i \in I} N \bigotimes M_i$ by Proposition 2.14 iii), we can take a map $j_i : N \bigotimes_A M_i \to N' \bigotimes_A M_i$ by $j_i = \pi_{N' \bigotimes_A M,i} \circ j \otimes 1 \circ \iota_{N \bigotimes_A M,i}$. Notes also that $\pi_{N' \bigotimes_A M,j} \circ j \otimes 1 \circ \iota_{N \bigotimes_A M,i} = 0$ since its M part goes zero. Thus this construction satisfies above claim. Hence, M is flat if and only if $j \times 1$ is injective if and only if all j_is are injective by above claim if and only if M_i is flat since j was arbitrarily chosen.

For later use, we claim that

Claim IX. *Free A-module is flat A-module.*

Proof. Free module is direct sum of A as an A-module. And $A \bigotimes_A (-)$ is an exact functor since $A \bigotimes_A M \cong M$ for any M. Thus, A is flat. Therefore any direct sum of copies of A should be flat by Exercise 2.4 ☐

5. As an A-module, $A[x] \cong \bigoplus_{i \in \mathbb{N}} M_i$ where $M_i = (x^i)$. To see this, let $\phi : A[x] \to \bigoplus_{i \in \mathbb{N}} M_i$ by sending x^i to $x^i \in M_i$. By construction of each elements, it is definitely additive homomorphism. Also, for any $f = a_0 + a_1 x + \cdots + a_k x^k \in A[x]$, $\phi(af) = (aa_0, \cdots, aa_n, 0, \cdots) = a(a_0, \cdots, a_n, 0, \cdots) = a\phi(f)$. Hence ϕ is module map, and it is clear that ϕ is bijective. Hence isomorphism.

Now notes that each M_i is free A-module, i.e., $(x^i) \cong A$ as an A-module. Thus by Claim IX, it is flat. Thus, by Exercise 4, $A[x]$ is flat A-module.

6. Define an action of $A[x]$ on $M[x]$ as for any $(\sum_{i=0}^{n} a_i x^i) \in A[x]$ and $(\sum_{j=0}^{m} m_i x^i) \in M[x]$,

$$(\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{m} m_i x^i) = \sum_{k=0}^{m+n} c_k x^k$$

where $c_k = \sum_{j=0}^{k} a_j m_{k-j}$, and if we regard $a_l, m_{l'}$ as zero for $l \notin [n], l' \notin [m]$. Notes that multiplication am from $a \in A$ and $m \in M$ denotes action of A on M. To check that it gives a module structure, we need to show that distributivity and associativity holds. For associativity,

$$\begin{aligned}
(\sum_{i=0}^{n} a_i x^i + \sum_{l=0}^{n'} a'_i x^i)(\sum_{j=0}^{m} m_i x^i) &= \sum_{k=0}^{\max(n,n')+m} \left(\sum_{j=0}^{k} (a_j + a'_j) m_{k-j} \right) x^k \\
&= \sum_{k=0}^{\max(n,n')+m} \left(\sum_{j=0}^{k} a_j m_{k-j} \right) x^k + \sum_{k=0}^{\max(n,n')+m} \left(\sum_{j=0}^{k} a'_j m_{k-j} \right) x^k \\
&= \sum_{k=0}^{n+m} \left(\sum_{j=0}^{k} a_j m_{k-j} \right) x^k + \sum_{k=0}^{n'+m} \left(\sum_{j=0}^{k} a'_j m_{k-j} \right) x^k \\
&= (\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{m} m_i x^i) + (\sum_{l=0}^{n'} a'_i x^i)(\sum_{j=0}^{m} m_i x^i).
\end{aligned}$$

And the other way is similar. For distributivity, it is usual product on polynomial ring, and just multiplication of coefficient changed to module action, so it holds. Also $1 \in A[x]$ works as identity. Hence $M[x]$ is a $A[x]$-module. Now notes that $M[x] \cong \bigoplus_{i\in\mathbb{N}} M'_i$ where $M'_i = x^i M$. (Showing isomorphism is exactly the same as we did in the proof of Exercise 5.) Hence, if we use notation $A[x] \cong \bigoplus_{i\in\mathbb{N}} M_i$ in the proof of Exercise 5,

$$A[x] \bigotimes_A M \cong \bigoplus_{i\in\mathbb{N}} M_i \bigotimes_A M = \bigoplus_{i\in\mathbb{N}} x^i A \bigotimes_A M \cong \bigoplus_{i\in\mathbb{N}} x^i M \cong M[x],$$

as an A-module. Notes that $x^i A \bigotimes_A M \cong x^i M$ comes from the fact that each $x^i A \cong A$ as an A-module, thus $x^i A \bigotimes_A M \cong A \bigotimes_A M \cong M \cong x^i M$ as an A-module.

7. Let $\phi : A[x] \to (A/\mathfrak{p})[x]$ by $a_i x^i \mapsto \bar{a}_i x^i$ for any $i \in \mathbb{N}$. It is homomorphism by construction. Also, it is surjective, since $A \to A/\mathfrak{p}$ is surjective. Thus, $\ker(\phi) \supseteq \mathfrak{p}[x]$. Conversely, if $f = \sum_{i=0}^{n} a_i x^i$ is in $\ker(\phi)$, then each a_i is in $\mathfrak{p}$, thus $f \in \mathfrak{p}[x]$. Since $A/\mathfrak{p}$ is an integral domain, its polynomial ring is integral domain, thus $\ker(\phi) = \mathfrak{p}[x]$ is prime ideal in $A[x]$.

Claim X. *VI Polynomial ring over integral domain is integral domain.*

Proof. Let A be an integral domain. Then, by Exercise 1.2 iii), $f \in A[x]$ is zero divisor if and only if there exists $a \in A$ such that $af = 0$. If f is nonzero zero divisor, then take a zero divisor f of the smallest degree. Then $\deg(f) >= 0$. (We use convention that $\deg(0) = -\infty$ in Lang's book.) Then, by mentioned exercise, $\exists a \in A$ such that $af = 0$. However, since A is integral domain, af doesn't change degree of f since a is nonzero. Thus, $-\infty = \deg(0) = \deg(af) = \deg(f) \geq 0$, contradiction. □

However, $\mathfrak{m}[x]$ is not a maximal ideal in general. Think about $\mathbb{Z}[x]$. Let $\mathfrak{m} = (2)$. Then, $\mathfrak{m}[x]$ is a set of polynomials whose coefficients are even. So, $\mathfrak{m}[x] = 2\mathbb{Z}[x]$. However, notes that $(2, x)$ from Exercise 1.16 properly contains $2\mathbb{Z}[x]$, since $x \notin 2\mathbb{Z}[x]$. So it is not a maximal.

8. (a) Suppose M, N are flat. Fix a short exact sequence $0 \to B' \to B \to B'' \to 0$. Then $0 \to B' \bigotimes_A M \to B \bigotimes_A M \to B'' \bigotimes_A M \to 0$ is exact by flatness of M with Proposition 2.15. By flatness of N with

Proposition 2.15, $0 \to (B' \bigotimes_A M) \bigotimes_A N \to (B \bigotimes_A M) \bigotimes_A N \to (B'' \bigotimes_A M) \bigotimes_A N \to 0$ is exact. By proposition 2.14 ii), $0 \to B' \bigotimes_A (M \bigotimes_A N) \to B \bigotimes_A (M \bigotimes_A N) \to B'' \bigotimes_A (M \bigotimes_A N) \to 0$ is exact. Since $0 \to B' \to B \to B'' \to 0$ was chosen arbitrarily, by Proposition 2.15, $M \bigotimes_A N$ is flat.

(b) Let $j : M \to M'$ be injective A-module map. Let $f : A \to B$ be a ring map making B as an A-algebra. First of all, since B is flat A-algebra, which implies flat as an A-module,

$$j \times 1 : M \bigotimes_A B \to M' \bigotimes_A B$$

is injective map.

Also, since N is flat as B-algebra, and $j \times 1$ can be regarded as a B-module injective map. To see this, since B itself is (A, B)-bimodule, $M \bigotimes_A B$ and $M' \bigotimes_A B$ are naturally a B-module by Exercise 2.15, and the map $j \times 1$ is B-module homomorphism, since it is additive homomorphism inherited from A-module homomorphism and for any $b \in B, m \otimes b' \in N \bigotimes_A B$,

$$b.j \otimes 1(m \otimes b') = b.(j(m) \otimes b') = j(m) \otimes bb' = j \otimes 1(m \otimes bb') = j \otimes 1(b.(m \otimes b')).$$

Thus, since N is a flat B-module, by Proposition 2.15,

$$j \times 1 : (M \bigotimes_A B) \bigotimes_B N \to (M' \bigotimes_A B) \bigotimes_B N$$

is injective. Using Exercise 2.15, we have a map

$$M \bigotimes_A N \cong M \bigotimes_A (B \bigotimes_B N) \cong (M \bigotimes_A B) \bigotimes_B N \to (M' \bigotimes_A B) \bigotimes_B N \cong M' \bigotimes_A (B \bigotimes_B N) \cong M' \bigotimes_A N.$$

Since this map is coposition of isomorphisms and injective maps, thus it is injective. Hence N is a flat A-module.

9. Notes that $M'' \cong M/\ker(M \to M'')$, and $\ker(M \to M'') \cong M'$ as an A-module. Hence we can regard this short exact sequence as $0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M/M' \to 0$. Now let $\{x_i\}_{i \in I}, \{\overline{y_j}\}_{j \in J}$ are finite generating set of M' and M/M'. Then, we can regard y_j as lifting of $\overline{y_j}$. Thus a submodule N of M generated by $\{x_i, y_j\}_{i \in I, j \in J}$ is finitely generated module. Also, N contains M' and $\pi(N) = M/M'$. Thus, by a one-to-one order preserving correspondence between submodules of M containing M' and submodules of M/M', this implies that N is the largest submodule of M, which is M itself. Hence M is finitely generated.

10. Think about the map $M \twoheadrightarrow M/\mathfrak{a}M \twoheadrightarrow N/\mathfrak{a}N$ by $m \mapsto m + \mathfrak{a}M \mapsto u(m) + \mathfrak{a}N$. Then, from the surjectivity, $u(M) + \mathfrak{a}N = N$. Then by Corollary 2.7, $u(M) = N$. This implies u is surjective.

11. For the first question, let $\mathfrak{m}$ be a maximal ideal of A, and let $\phi : A^m \to A^n$ be an isomorphism as A-module. Then, $1 \otimes \phi : (A/\mathfrak{m}) \bigotimes_A A^m \to (A/\mathfrak{m}) \bigotimes_A A^n$ is also an isomorphism as A-module. (To see this, we can use Proposition 2.18 with an exact sequence $0 \to A^m \xrightarrow{\phi} A^n \to 0$.) Then, by Proposition 2.14, if we let $k = A/\mathfrak{m}$, a residue field, then $(A/\mathfrak{m}) \bigotimes_A A^m \cong (A/\mathfrak{m})^m = \bigoplus_{i=1}^m k \cong k^m$ and by the same argument $(A/\mathfrak{m}) \bigotimes_A A^n \cong k^n$. Also, if $a, b \in A$ such that $a = b + m$ for some $m \in \mathfrak{m}$, then for any $\overline{x} \in A/\mathfrak{m}, (a_1, \cdots, a_m) \in A^m$,

$$\begin{aligned} a.(\overline{x} \otimes (a_1, \cdots, a_m)) &= (b + m).(\overline{ax} \otimes (a_1, \cdots, a_m)) = (\overline{bx} \otimes (a_1, \cdots, a_m)) + (\overline{mx}, (a_1, \cdots, a_m)) \\ &= (\overline{bx} \otimes (a_1, \cdots, a_m)) = b.(\overline{x} \otimes (a_1, \cdots, a_m)). \end{aligned}$$

Thus we can regard $(A/\mathfrak{m})\bigotimes_A A^m \cong k^n$ as a k-module. Hence it has a vector space structure. Also, the map $1\otimes j$ is also k-module map, since for any $a,b\in A$ such that $a=b+m$ for some $m\in\mathfrak{m}$,

$$a.1\otimes j(\overline{x}\otimes(a_1,\cdots,a_m)) = j(a.(\overline{x}\otimes(a_1,\cdots,a_m))) = j(b.(\overline{x}\otimes(a_1,\cdots,a_m))) = b.1\otimes j(\overline{x}\otimes(a_1,\cdots,a_m)).$$

Hence, $1\otimes j$ is k-module map, which implies that it is just a vector space map. And we already know that $k^n\cong k^m$ as a vector space if and only if $n=m$, using argument on the basis. Done.

For the second question, if $\phi: A^m\to A^n$ is surjective, so does $1\otimes j$. (We can use Proposition 2.18 with an exact sequence $\ker\phi\to A^m\xrightarrow{f} A^n\to 0$.) And by the above argument, since $1\otimes j$ is a vector space homomorphism, this implies $1\otimes j(A^m)=A^n$ Since $\dim(1\otimes j(A^m))\le m$, this implies that $n\le m$.

For the third question, we can use Proposition 2.4. We cannot use the same argument since tensor is right exact.

Suppose $\phi: A^m\to A^n$ is injective but $m>n$. Then, we can regard A^n as a submodule of A^m, by stating that $A^n=\{(a_1,\cdots,a_n,0,\cdot,0)\in A^m : a_i\in A\}$. Then, clearly, if we let $\mathfrak{a}=A$, then $\phi(A^m)\subseteq A^n\subseteq A^m=\mathfrak{a}A^m$. Also, notes that A^m is finitely generated A-module. Then, Proposition 2.4 states that there exists an equation of the form

$$f(\phi) := \phi^n + a_1\phi^{n-1}+\cdots+a_n = 0$$

where the a_i are in $\mathfrak{a}=A$. If this polynomial has minimal possible degree, then $a_n\neq 0$; if $a_n=0$, then $f(\phi)=\phi(g(\phi))$ for some polynomial g of ϕ, hence $g(\phi)=0$ since injectivity of ϕ implies that ϕ is nonzero for all values $a\in A\setminus\{0\}$. Now notes that $\phi(v)\in A^n$ for any $v\in A^m$, thus $\phi(v)=(v_1,\cdots,v_n,0,\cdots,0)$ for all $v\in A^m$. However,

$$\begin{aligned} f(\phi)(0,\cdots,0,1) &= (\phi^n + a_1\phi^{n-1}+\cdots a_{n-1}\phi)(0,\cdots,0,1) + a_n(0,\cdots,0,1)\\ &= (a_1,\cdots,a_n,0,\cdots,0)+(0,\cdots,0,a_n) = (0,\cdots,0). \end{aligned}$$

This implies $a_n=0$, contradiction.

12. We need another claim for direct sum of module.

Claim XI. *VII If M,N are A-modules such that there exists a module map $r: M\to N$ and $s: N\to M$ such that $r\circ s = 1_N$, then $M\cong N\oplus\operatorname{coker}(s)\cong N\oplus\ker(r)$.*

Proof. Let $\phi: M\to N\oplus\operatorname{coker}(s)$ by $m\mapsto(r(m),\overline{m})$ (since $\operatorname{coker}(s)=M/\operatorname{Im}(s)$.) If $m\in\ker(\phi)$, then $r(m)=0, m\in\operatorname{Im}(s)$. Hence $\exists n\in N$ such that $s(n)=m$ and $0=r(m)=r\circ s(n)=n$. Hence $m=0$. This implies injectivity. To see surjectivity, let $y\in N,\overline{m}\in\operatorname{coker}(s)$. Then, let $z=m+s(y-r(m))$, then

$$\phi(z) = (r(m+s(y-r(m))),\overline{m+s(y-r(m))}) = (r(m)+r\circ s(y-r(m)),\overline{m}) = (r(m)+y-r(m),\overline{m}) = (y,\overline{m}).$$

Hence, it is surjective map. So isomophism.

Also, take a map $\psi:\operatorname{coker}(s)\to\ker(r)$ by $\overline{m}\mapsto m - s\circ r(m)$. First of all, it is well-defined, since for any two representative $m,m'\in\overline{m}$, $m=m'+s(n)$ for some $n\in N$, hence

$$\psi(m) = m - s\circ r(m) = m'+s(n)-s\circ r(m'+s(n)) = m'+s(n)-s\circ r(m')-s(n) = m'-s\circ r(m') = \psi(m').$$

Also, $r(m-s\circ r(m)) = r(m)-r\circ s\circ r(m) = r(m)-r(m)=0$. Hence this map is well-defined. Moreover, it is injective since for $\psi(\overline{m})=0$, $m=s\circ r(m)$. However, $\overline{m}\neq 0$ if and only if $m\notin\operatorname{Im}(s)$. Thus the only possible m satisfying $m=s\circ r(m)$ is when $m=0$. Also, it is surjective, since for any $m\in\ker(r)$, $m-s\circ r(m)=m$. Hence $\psi(\overline{m})=m$, done. □

Let $e_1,\cdots,e_n$ be a basis of A^n. Then from surjectivity $\exists u_i\in M$ such that $\phi(u_i)=e_i$. Then, let N be a submodule of M generated by $u_1,\cdots,u_n$. Also define $\psi: A^n\to N\subseteq M$ by $e_i\mapsto u_i$. Then, $\phi\circ\psi = 1_{A^n}$. By above claim, $M=N\oplus\ker(\phi)$ with $N\cong A^n$. To show that $\ker(\phi)$ is finitely generated, notes that $M/N\cong\ker(\phi)$. Then, let $\{x_i\}_{i=1}^m$ be a generating set of M. Then, $\{x_i+N\}_{i=1}^m$ generates M/N since every element in M/N have a representation which is linear combinationf of x_is. Thus $\ker(\phi)$ is finitely generated.

13. Define $p : N_B \to N$ by $p(b \otimes y) = by$. Then, $p \circ g(y) = p(1 \otimes y) = y$. Thus $p \circ g = 1_N$. This implies g is injective since composition is injective if and only if the rightmost term is injective. Also, if we define $p' : N_B \to g(N)$ by $(b \otimes y) = 1 \otimes by$, and $g' : g(N) \to N_B$ by $1 \otimes n \mapsto 1 \otimes n$, then $p' \circ g' = 1_{g(N)}$. Hence by the above claim,
$$N_B \cong g(N) \oplus \ker(p')$$
Hence $g(N)$ is direct summands of N_B. And in this case, $b \otimes y$ corresponds to $(1 \otimes by, b \otimes y - 1 \otimes by)$.

14. To see $\mu_i = \mu_j \circ \mu_{ij}$ for $i \leq j$, let $x_i \in M_i$. Then, $x_i - \mu_{ij}(x_i) \in D = \ker(\mu)$. Hence,
$$\mu_i(x_i) = \mu(x_i) = \mu(x_i + \ker(\mu)) = \mu(\mu_{ij}(x_i) + \ker(\mu)) = \mu(\mu_{ij}(x_i)) = \mu_j(\mu_{ij}(x_i)).$$
Notes that first and last equality comes from the restriction.

15. Let $x \in M$. Then, x has a representative in $C = \bigoplus_{i \in I} M_i$. Since it is direct sum x is tuple of finitely many nonzero elements, say $x = \sum_{j \in J} x_j$, for $|J| < \infty$. Thus by applying directed set property $|J| - 1$ times, we can get $i \in I$ such that $i \geq j$ for all $j \in J$. Now let $\tilde{x} = \sum_{j \in J} \mu_{ji}(x_j)$. Then,
$$x = \mu\left(\sum_{j \in J} x_j\right) = \sum_{j \in J} \mu(x_j) = \sum_{j \in J} \mu_j(x_j) = \sum_{j \in J} \mu_i \circ \mu_{ji}(x_j) = \mu i\left(\sum_{j \in J} \mu_{ji}(x_j)\right).$$
Thus if we let $x_i = \sum_{j \in J} \mu_{ji}(x_j)$, then $x_i \in M_i$, and $\mu_i(x_i) = x \in M$.
For the second statement, suppose $\mu_i(x_i) = 0$ for some $i \in I$. Then, $x_i \in M_i \cap D$. Hence, x_i is finite sum of generators of D in M_i, i.e.,
$$x_i = \sum_{j,k \in T \subseteq J^2} (y_j - \mu_{jk}(y_j)).$$
for some finite subset J of I, and $y_j \in M_j$ for each $j \in J$. Now by the same argument in the above proof, there exists $l \in I$ such that $l \geq k$ for all $k \in J \cup \{i\}$. Also, we can apply μ_{il} on each y_j in the summands as μ_{jl} since $y_j = 0$ if $j \neq i$, and also apply it on $\mu_{jk}(y_j)$ as μ_{kl} since $\mu_{jk}(y_j) = 0$ if $k \neq i$. Hence,
$$\mu_{il}(x_i) = \sum_{j,k \in T \subseteq J^2} \mu_{il}(y_j - \mu_{jk}(y_j)) = \sum_{j,k \in T \subseteq J^2} (\mu_{jl}(y_j) - \mu_{kl} \circ \mu_{jk}(y_j)) = 0$$
where the last equality comes from $\mu_{jl} = \mu_{jk}\mu_{kl}$.

16. We use notation of Exercise 2.14. From the universal property of direct product, $\exists \tilde{\alpha} : C = \bigoplus_{i \in I} M_i \to N$ such that for each $x_i \in M_i$, $\tilde{\alpha} \circ \iota_j = \alpha_j$. Now check that for any generator $x_i - \mu_{ij}(x_i) \in D$,
$$\tilde{\alpha}(x_i - \mu_{ij}(x_i)) = \alpha_i(x_i) - \alpha_j \circ \mu_{ij}(x_i) = \alpha_j \circ \mu_{ij}(x_i - x_i) = 0$$
where the last equaltiy comes from the given condition $\alpha_j \circ \mu_{ij} = \alpha_i$. Hence we can lift this map on $C/D = \varinjlim M_i$. Denote $\alpha : \varinjlim M_i \to N$ be the lifting map. Then, for any $x_i \in M_i$,
$$\alpha \circ \mu_i(x_i) = \tilde{\alpha}(x_i) = \alpha_i(x_i).$$
To see the homomorphism is unique, let $\alpha' : M \to N$ be another homomorphism satisfying $\alpha_i = \alpha' \circ \mu_i$. Then, by Exercise 15, let $\mu_i(x_i)$ be an arbitrary elements of M_i. Then, $\alpha'(\mu_i(x_i) = \alpha_i = \alpha \circ \mu_i(x_i)$. Hence $\alpha' = \alpha$.

Notes that this exercise is about universal property of direct limit. Now for later use, we claim that direct limit itself is unique, i.e., a module M in Exercise 16 is unique.

Claim XII. *VIII Direct limit is unique, in the sense that if there exists $(M', \mu_i' : M_i \to M')$ such that for given N and $\alpha_i : M_i \to N$, there exists $\alpha' : M' \to N$ such that $\alpha' \circ \mu_i = \alpha_i$, then $M' = M$ and $\mu_i' = \mu_i$.*

Proof. What we've shown can be denoted as below diagram.

$$\begin{tikzcd}
M_i \arrow[rr, "\mu_{ij}"] \arrow[dr, "\mu_i"] \arrow[ddr, "\alpha_i"'] & & M_j \arrow[dl, "\mu_j"'] \arrow[ddl, "\alpha_j"] \\
& \varinjlim M_i = M \arrow[d, dashed, "\exists!\alpha"] & \\
& N &
\end{tikzcd}$$

If we have $(M', \mu_i' : M_i \to M')$ be another direct limit, i.e., satisfying properties of M in Exercise 16, then by letting $N := M'$ and $\alpha_i := \mu_i'$, we have a unique map $\beta : \varinjlim M_i \to M'$ such that $\beta \circ \mu_i = \mu_i'$. Conversely, since M' acting like M, if we let $N = M = \varinjlim M_i$ and $\alpha_i = \mu_i$ in viewpoint of N, the by the Exercise 16, there exists $\gamma : M' \to \varinjlim M_i$ such that $\gamma \circ \mu_i' = \iota_i$. Hence,

$$\beta \circ \gamma \circ \mu_i' = \mu_i', \gamma \circ \beta \circ \mu_i = \mu_i$$

for all $i \in I$. For the right one, since every elements in $\varinjlim M_i$ can be denoted as $\mu_i(x_i)$ for some $i \in I$, this implies $\gamma \circ \beta = 1_{\varinjlim M_i}$. For the left one, notes that if we let $M = M'$ and $N = M'$, $\alpha_i = \mu_i'$, then the above universal property gives us unique map $\xi : M' \to M'$ satisfying $\xi \circ \mu_i' = \mu_i'$, and since $1_{M'}$ also has property that $1_{M'} \circ \mu_i' = \mu_i'$, thus $\xi = 1_{M'}$. Now the left one implies that $\beta \circ \gamma$ also the map satisfying $\beta \circ \gamma \circ \mu_i' = \mu_i'$, hence $\beta \circ \gamma = 1_{M'}$. (Notes that this argument is more categorical than the previous one using Exercise 15.) Hence, β and γ are inverse to each other, thus $M' \cong \varinjlim M_i$, done. ☐

17. First of all, $\sum M_i \supseteq \bigcup M_i$ since each $M_i \subseteq \sum M_i$. On the other hand, if $x \in \sum M_i$, $x = \sum_{j \in J} x_j$ for some finite index set J of directed set I. Hence there exists $k \in I$ such that $j \leq k$ for all $j \in J$. Thus $x \in M_k$, which implies $x \in \bigcup M_i$.Thus $\sum M_i = \bigcup M_i$ and $\bigcup M_i$ has a module structure.

    Now to see $\varinjlim M_i = \bigcup M_i$ as a module, if we regard M_i as canonical subset of $\varinjlim M_i$, then $\varinjlim M_i \supseteq \bigcup M_i$. To see equality, we use the uniqueness of direct limit. Let N be an A-module having maps in Exercise 16. Then, construct $\alpha : \bigcup M_i \to N$ by $x_i \mapsto \alpha_i(x_i)$. Then, if we let $\mu_i : M_i \to \bigcup M_i$ as canonical injection, then $\alpha \circ \mu_i = \alpha_i$. Since N was arbitrary, $\bigcup M_i$ is also direct limit, thus by uniqueness, it is isomorphic to $\varinjlim M_i$. Since we already identify $\bigcup M_i$ as subset of $\varinjlim M_i$, we can say that they are equivalent.

    To see the last statement, for any $x \in M$, $x \in Ax \subseteq M$, thus M is union of finitely generated submodules. Also, this index of finitely generated submodules is directed, since for any two finitely generated submodule, their sum is also a finitely generated submodule. Hence with canonical inclusion maps, it comprise a directed system, and by the statement of this Exercise, $M = \varinjlim M_i$.

18. From $\nu_i \circ \phi_i : M_i \to N$, from the universal property of direct limit (in Exercise 16) we have a map $\phi : M \to N$ such that $\nu_i \circ \phi_i = \phi \circ \mu_i$.

19. Let $\phi : M \to N$ and $\psi : N \to P$ be a module map induced by homomorphism of directed system. Let $\mathbf{M} = (M_i, \mu_{ij}), \mathbf{N} = (N_i, \nu_{ij}), \mathbf{P} = (P_i, \xi_{ij})$. Also, $\mu_i : M_i \to M, \nu_i : N_i \to N, \xi_i : P_i \to P$. Then, $\psi_i \circ \phi_i = 0$ by exactness of $\mathbf{M} \to \mathbf{N} \to \mathbf{P}$. Thus,

    $$0 = \xi_i \circ \psi_i \circ \phi_i = \psi \circ \phi \circ \mu_i.$$

    Since $\psi \circ \phi$ are induced by the universal property, so it is unique, and 0 also satisfy the condition of universal property, this implies $\psi \circ \phi = 0$. Hence $\mathrm{Im}(\phi) \subseteq \ker(\phi)$.

    To see they are equal, let $n \in \ker(\phi)$. Then, by Exercise 15, $n = \nu_i(n_i)$ for some i and $n_i \in N_i$. Then, since

    $$0 = \psi(n) = \psi \circ \nu_i(n_i) = \xi_i \circ \psi_i(n_i)$$

and ξ_i is injective, $n_i \in \ker(\psi_i) = \mathrm{Im}(\phi_i)$. Hence $n_i = \phi_i(m_i)$ for some m_i, therefore,

$$n = \nu_i(n_i) = \nu_i \circ \phi_i(m_i) = \phi \circ \mu_i(m_i).$$

This implies $n \in \mathrm{Im}(\phi)$, done.

20. Let $g_i : M_i \times N \to M_i \bigotimes_A N$ be the canonical bilinear mapping defined when we construct the tensor product $M_i \times N$. Then, let $P = \varinjlim(M_i \bigotimes_A N)$, and $R = \varinjlim M_i \times N$. Notes that $R \cong \varinjlim M_i \times N$; to see this, let $\alpha_i : M_i \times N \to Q$ for some A-module Q such that for all $i \leq j$, $\alpha_i = \alpha_j \circ (\mu_{ij} \times 1_N)$. Now define $\alpha : \varinjlim M_i \times N \to Q$ by $(\mu_i(x_i), n) \mapsto \alpha_i(x_i, n)$. Then, this satisfies $\alpha \circ (\mu_i \times 1_N)(x_i, n) = \alpha(\mu_i(x_i), n) = \alpha_i(x_i, n)$. Thus, by the uniqueness of direct limit, $R \cong \varinjlim M_i \times N = M \times N$.

    Now to see that $P \cong M \times N$, notes that g_i induces a homomorphism of direct system. To check this, we already show that $(M \times N, \mu_{ij} \times 1)$ is also a direct system. Hence,

$$\mu_{ij} \otimes 1_N \circ g_i(m_i, n) = \mu_{ij} \otimes 1_N(m_i \otimes n) = \mu_{ij}(m_i) \otimes n$$

    and

$$g_j \circ \mu_{ij} \times 1_N(m_i, n) = g_j(\mu_{ij}(m_i), n) = \mu_{ij}(m_i) \otimes n.$$

    Therefore, by Exercise 18, it induces a homomorphism $g : M \times N \to P$. Now notes that g is A-bilinear. To see this, by Exercise 15, every element in $M \times N$ can be denoted as $\mu_i(m_i) \times n$, thus if we fix $n \in N$, then

$$\begin{aligned}
g((a_i\mu_i(m_i), n) + (a_j\mu_j(m_j), n)) &= g((\mu_i(a_im_i), n) + \mu_j(a_jm_j), n)) = g((\mu_i(a_im_i), n)) + g(\mu_j(a_jm_j), n)) \\
&= g \circ \mu_i \times 1_N(a_im_i, n) + g \circ \mu_j \times 1_N(a_jm_j, n) \\
&= \mu_i \otimes 1_N \circ g_i(a_im_i, n) + \mu_j \otimes 1_N \circ g_j(a_jm_j, n) \\
&= a_i\mu_i(m_i) \otimes n + a_j\mu_j(m_j) \otimes n \\
&= a_ig(\mu_i(m_i), n) + a_jg(\mu_j(m_j), n).
\end{aligned}$$

    Also, if we fix M part, then linearity also holds since $a \otimes n + b \otimes n = (a+b) \otimes n$. Thus, by the universal property of tensor product, we have a map $\phi : M \otimes N \to P$. We already have $\psi : P \to M \otimes N$ using Exercise 16. To see they are inverse, fix an arbitrary elements of P as $\mu_i \otimes 1_N(m_i, n)$ by Exercise 15. Then,

$$\phi \circ \psi(\mu_i \otimes 1_N(m_i, n)) = \phi \circ \psi_i(m_i \otimes n) = \phi(\mu_i(m_i) \otimes n) = g(\mu_i(m_i), n) = \mu_i(m_i) \otimes n = \mu_i \otimes 1_N(m_i, n).$$

    and for arbitrary element $\mu_i(m_i) \otimes n$ in $M \times N$ by Exercise 15,

$$\psi \circ \phi(\mu_i(m_i) \otimes n) = \psi \circ g(\mu_i(m_i), n) = \psi(\mu_i(m_i) \otimes n) = \psi_i(m_i \otimes n) = \mu_i(m_i) \otimes n.$$

    Hence they are inverse to each other.

21. First of all, we will show that A is a ring. Let $a, b \in A$. By exercise 2.15, $a = \alpha_i(x_i)$, $b = \alpha_j(x_j)$ for some $i, j \in I$. Since I is direct set, $\exists k \in I$ such that $i \leq k, j \leq k$. Hence, from $x_i - \alpha_{ik}(x_i)$, we can get

$$\alpha_i(x_i) = \alpha(x_i) = \alpha(\alpha_{ik}(x_i) = \alpha_k(\alpha_{ik}(x_i))$$

    since $\alpha(x_i - \alpha_{ik}(x_i) = 0$ is projection. Likewise, $\alpha_j(X_j) = \alpha_k(\alpha_{jk}(x_j)$. Hence, define

$$a \cdot b := \alpha_k(\alpha_{ik}(x_i) \cdot \alpha_{jk}(x_j)).$$

    Since $\alpha_{ik}(x_i), \alpha_{jk}(x_j) \in A_k$, this definition is possible. Also, to see it is well-defined, choose another k' which is distinct to k. Then from the definition of directed set, $\exists l \in I$ such that $k', k \leq l$, hence

$$\alpha_k(\alpha_{ik}(x_i) \cdot \alpha_{jk}(x_j)) = \alpha_l(\alpha_{kl}(\alpha_{ik}(x_i) \cdot \alpha_{jk}(x_j))) = \alpha_l(\alpha_{kl}(\alpha_{ik}(x_i) \cdot \alpha_{jk}(x_j))) = \alpha_l(\alpha_{il}(x_i) \cdot \alpha_{jl}(x_j))$$

and

$$\alpha_{k'}(\alpha_{ik'}(x_i)\cdot\alpha_{jk'}(x_j)) = \alpha_l(\alpha_{k'l}(\alpha_{ik'}(x_i)\cdot\alpha_{jk'}(x_j))) = \alpha_l(\alpha_{k'l}(\alpha_{ik'}(x_i)\cdot\alpha_{jk'}(x_j))) = \alpha_l(\alpha_{il}(x_i)\cdot\alpha_{jl}(x_j)).$$

Hence this definition is independent of choice of k. Also, this definition is independent of choice of representation of b. To see this, let $b = \alpha_{j'}(x_{j'})$ for another j'. Then, let $k \geq i, j, j'$, thus

$$a\cdot\alpha_j(x_j) - a\cdot\alpha_{j'}(x_{j'}) = \alpha_k(\alpha_{ik}(x_i)\cdot\alpha_{jk}(x_j)) - \alpha_k(\alpha_{ik}(x_i)\cdot\alpha_{j'k}(x_{j'})) = \alpha_k(\alpha_{ik}(x_i)\,(\alpha_{jk}(x_j) - \alpha_{j'k}(x_{j'})))$$

thus it suffices to show that $\alpha_{jk}(x_j) - \alpha_{j'k}(x_{j'}) = 0$. To see this, let $c = \alpha_{jk}(x_j) - \alpha_{j'k}(x_{j'})$. Then, $\alpha_k(c) = 0$ since they are the same representation of b on the direct limit. By Exercise 2.15, $\exists l \geq k$ such that $\alpha_{kl}(c) = 0$. This implies

$$0 = \alpha_{kl}(\alpha_{jk}(x_j) - \alpha_{j'k}(x_{j'})) = \alpha_{jl}(x_j) - \alpha_{j'l}(x_{j'}) \implies \alpha_{jl}(x_j) = \alpha_{j'l}(x_{j'}).$$

Thus,

$$\begin{aligned} a\cdot\alpha_j(x_j) - a\cdot\alpha_{j'}(x_{j'}) &= \alpha_k(\alpha_{ik}(x_i)\,(\alpha_{jk}(x_j) - \alpha_{j'k}(x_{j'}))) = \alpha_l\circ\alpha_{kl}(\alpha_{ik}(x_i)\,(\alpha_{jk}(x_j) - \alpha_{j'k}(x_{j'}))) \\ &= \alpha_l(\alpha_{il}(x_i)\,(\alpha_{jl}(x_j) - \alpha_{j'l}(x_{j'}))) = \alpha_l(0) = 0. \end{aligned}$$

Thus product is well-defined. Hence A has a ring structure.

Also, for any $a, b \in A_i$,

$$\alpha_i(ab) = \alpha_i(\alpha_{ii}(a)\alpha_{ii}(b)) = \alpha_i(a)\alpha_i(b)$$

by definition of the product in A. Hence, α_i is multiplicative homomorphism. Also, for any $b \in A$ with $b = \alpha_j(x_j)$ for some $j \in I$, there exists $k \geq i, j$ so that

$$\alpha_i(1)b = \alpha_k(\alpha_{ik}(1)\alpha_{jk}(x_j)) = \alpha_k(1\cdot\alpha_{jk}(x_j)) = \alpha_k\circ\alpha_{jk}(x_j) = b$$

where $\alpha_{ik}(1) = 1$ is from the fact that α_{ik} is a ring homomorphism. Hence, this homomorphism preserves 1. Thus α_i is a ring homomorphism.

To verify other axioms of the ring, for any given elements, choose some A_k that they lie and use axioms of the ring for A_k to get desired result. Now suppose $A = 0 \iff A_i = 0$ for some $i \in I$. If $A = 0$, then $1 = 0$, thus $\alpha_i(1) = 0$ for all $i \in I$. By exercise 2.15, for fixed i, there exists $j \in I$ such that $\alpha_{ij}(1) = 0$ Since α_{ij} is also a ring homomorphism from A_i to A_j, this implies that $1 = 0$ in A_j, thus $A_j = 0$. Conversely, if $A_i = 0$, then for any $j \geq i$, identity 1 in A_j should be $1 = \alpha_{ij}(1) = \alpha_{ij}(0) = 0$. Hence $A_j = 0$ for all $j \geq i$. Now for any $a \in A$, $a = \alpha_k(x_k)$, thus take $q \geq k, i$ so that

$$a = \alpha_k(x_k) = \alpha_q\circ\alpha_{kq}(x_k) = \alpha_q(0) = 0.$$

This implies $A = 0$.

22. Notes that each $\alpha_{ij}|_{\mathfrak{R}_i} : \mathfrak{R}_i \to \mathfrak{R}_j$ satisfies axioms in Exercise 14 as a $\mathbb{Z}$-module. Thus, we can construct $\varinjlim \mathfrak{R}$ as $\mathbb{Z}$-module. It is clear that $\varinjlim \mathfrak{R}$ is submodule of A.

Let a be a nilpotent element of A. Then $\exists n \in \mathbb{N}$ such that $a^n = 0$. By Exercise 2.15, $a = \alpha_i(x)$ thus $0 = a^n = \alpha_i(x^n)$. By Exercise 2.15, $\alpha_i(x^n) = 0$ implies $\exists j \geq i$ such that $\alpha_{ij}(x^n) = 0$. Hence, $\alpha_{ij}(x)$ is in the nilradical of A_j, and

$$\alpha_j|_{\mathfrak{R}_j}(\alpha_{ij}(x)) = \alpha_j(\alpha_{ij}(x)) = \alpha_i(x) = a$$

implies $a \in \varinjlim \mathfrak{R}$. Thus $\varinjlim \mathfrak{R}$ contains nilradical of A. Conversely, if $a \in \varinjlim \mathfrak{R}$, by Exercise 2.15, $a = \alpha_i(x_i)$ with $x_i \in \mathfrak{R}_i$ by construction. Thus $\exists n \in \mathbb{N}$ such that $x_i^n = 0$. This implies $a^n = \alpha_i(x_i^n) = \alpha_i(0) = 0$ since by Exercise 2.21, α_i is a ring homomorphism. Thus a is in nilradical of A. This shows that $\varinjlim \mathfrak{R}$ is nilradical of A.

Suppose that $\exists c, d \in A$ such that $cd = 0$ but $c \neq 0, d \neq 0$. Then, $c = \alpha_i(x_i), d = \alpha_j(x_j)$ with nonzero x_i. (Notes that from $c \neq 0, d \neq 0$, all of their representation are nonzero; otherwise $c = \alpha_j(0)$ for some representation, which implies $c = 0$, contradiction.) Then, there exists $k \in I$ such that $k \geq i, j$, thus their product is

$$ab = \alpha_k(\alpha_{ik}(x_i)\alpha_{jk}(x_j)) = 0.$$

By Exercise 2.15, there exists $l \geq k$ such that

$$\alpha_{kl}(\alpha_{ik}(x_i)\alpha_{jk}(x_j)) = 0.$$

This implies $\alpha_{il}(x_i)\alpha_{jl}(x_j) = 0$. However, since c, d are nonzero, so are $\alpha_{il}(x_i)$ and $\alpha_{jl}(x_j)$. Thus A_l is not integral domain.

Hence, its contrapositive gives us that if each A_i is integral domain, so does A.

23. We already knows that a collection of all finite subsets of Λ form a direct set structure. With the canonical A-algebra homomorphism $\lambda_{JJ'} : B_J \to B_{J'}$ if $J \subseteq J'$ which is given by $b_j \otimes \cdots \otimes b_{j'} \mapsto b_j \otimes \cdots \otimes b_{j'} \otimes 1 \otimes \cdots \otimes 1$, the given direct limit B forms a ring by Exercise 21. To see that its A-algebra structure, notes that we have $A \to C \to B$ where C is direct sum of A-algebras, and $C \to B$ is a projection. Thus if $b \in B$, then $b = \alpha_J(x)$ for some $x \in B_J$. Now we can define A-action as

$$a.b = \alpha_J(ax).$$

This is well-defined action since for any other representation $b = \alpha_{J'}(y)$, we have $\alpha_J(x) = \alpha_{J\cup J'} \circ \alpha_{J,J\cup J'}(x)$ and $\alpha_{J'}(y) = \alpha_{J\cup J'} \circ \alpha_{J',J\cup J'}(y)$, therefore

$$0 = \alpha_J(x) - \alpha_{J'}(y) = \alpha_{J\cup J'} \circ (\alpha_{J,J\cup J'}(x) - \alpha_{J',J\cup J'}(y))$$

implies that (by Exercise 2.15) there exists $K \supseteq J \cup J'$ such that $\alpha_{J\cup J',K}(\alpha_{J,J\cup J'}(x) - \alpha_{J',J\cup J'}(y)) = 0$. This implies $\alpha_{JK}(x) - \alpha_{J'K}(y) = 0$. Thus,

$$\alpha_J(ax) = \alpha_K \circ \alpha_{JK}(ax) = \alpha_K(a\alpha_{JK}(x)) = \alpha_K(a\alpha_{J'K}(y)) = \alpha_K \circ \alpha_{J'K}(ay) = \alpha_{J'}(ay).$$

Thus, this action is independent of choice of representation, thus B is A-module. Also, we already know that α_J is a ring homomorphism. Moreover, this definition makes α_J be A-module homormophism structure, thus it is A-algebra map.

24. Notes that what the hint suggested follows from the balancing Tor theorem. In general $\operatorname{Tor}_n^A(M,N)$ is defined as a derived functor, i.e., applying $H_n(- \bigotimes_A N)$ on the given projective (or free) resolution. The balancing Tor theorem assures that for given any projective sequence $P^\bullet \to M$ and $Q^\bullet \to N$, $H_n(P^\bullet \bigotimes_A N) = H_n(M \bigotimes_A Q^\bullet)$. It seems that [3] intend the reader to use this theorem as a fact. So we will follow it.

$i) \to ii)$: Since M is flat, for any free resolution of N, say $Q^\bullet \to N$, $M \bigotimes Q^\bullet \to M \bigotimes N$ is also exact. (This is because every long exact sequence can be decomposed with short exact sequence, and M is flat implies that those all short exact sequence are exact.) Hence, $\operatorname{Tor}_n^A(M,N) = 0$ for any $n > 0$.

$ii) \to iii)$ is trivial.

$iii) \to i)$: Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence. Then, from the Tor exact seqeunce,

$$\operatorname{Tor}_1^A(M,N'') \to M \bigotimes N' \to M \bigotimes N \to M \bigotimes N'' \to 0$$

is exact. (Think about definition of left derived functor; it gives such a long exact seqeunce, and $\operatorname{Tor}_0^A(M,N) = M \bigotimes_A N$.) By iii), $\operatorname{Tor}_1^A(M,N'') = 0$. This gives one of the definition of flatness of M.

Finally, by the Balancing theorem, we can do the same thing with respect to when N is flat.

25. Suppose N' is flat. Then It gives an exact Tor sequence,

$$\operatorname{Tor}_2^A(M,N'') \to \operatorname{Tor}_1^A(M,N') \to \operatorname{Tor}_1^A(M,N) \to \operatorname{Tor}_1^A(M,N'') \to M \bigotimes N' \to M \bigotimes N \to M \bigotimes N'' \to 0$$

for any A-module M. Since N'' is flat, $\operatorname{Tor}_1^A(M,N'') = 0 = \operatorname{Tor}_2^A(M,N'')$. This gives us $\operatorname{Tor}_1^A(M,N') \cong \operatorname{Tor}_1^A(M,N)$. Hence, N' is flat if and only if $\operatorname{Tor}_1^A(M,N') = 0$ if and only if $\operatorname{Tor}_1^A(M,N) = 0$ if and only if N is flat by above exercise.

26. By Balancing Tor theorem and exercise 24, we already know that N is flat if and only if $\operatorname{Tor}_1(M,N) = 0$ for all A-module M. This implies that N is flat implies $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for all $\mathfrak{a}$ an ideal of A. Conversely, suppose $\operatorname{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideal $\mathfrak{a}$ of A. Suppose M is finitely generated, by $x_1, \cdots, x_n$. Let M_i be submodule of M generated by $x_1, \cdots, x_i$, with $M_0 = 0$. Then, $f_i : A \to M_i/M_{i-1}$ by $a \mapsto ax_i + M_{i-1}$ is surjective homomorphism, thus $\ker f_i$ is ideal, which implies $M_i/M_{i-1} \cong A/\mathfrak{a}_i$. for some finitely generated ideal $\mathfrak{a}_i$. Thus we have an exact sequence

$$0 \to M_{i-1} \to M_i \to A/\mathfrak{a}_i \to 0.$$

Notes that $\operatorname{Tor}_1(M_0, N) = \operatorname{Tor}_1(A/(1), N)) = 0$ by assumption. Suppose, to use induction, that $\operatorname{Tor}_1(M_{i-1}, 0) = 0$. Then, by long exact sequence of Tor,

$$\cdots \to \operatorname{Tor}_1(M_{i-1}, N) = 0 \to \operatorname{Tor}_1(M_i, N) \to \operatorname{Tor}_1(A/\mathfrak{a}_i, N) = 0 \to M_{i-1} \to M_i \to A/\mathfrak{a}_i \to 0$$

is exact sequence, which implies $\operatorname{Tor}_1(M_i, N) = 0$. From $M_n = M$, this implies that $\operatorname{Tor}_1(M,N) = 0$. Apply exercise 2.24, to get the result that N is flat.

27. Notes that idempotent ideal is an ideal $\mathfrak{a}$ such that $\mathfrak{a}^2 = \mathfrak{a}$.

i) $\to$ *ii)*: Let $x \in A$. Then, $A/(x)$ is a flat A-module, thus $(x) \to A$ is injective implies $\alpha : (x) \bigotimes_A A/(x) \to A \bigotimes_A A/(x) \cong A/(x)$ is injective. We know the exact map of α using Proposition 2.14, and it sends

$$\alpha; x \otimes \overline{a} \mapsto x \otimes \overline{a} \mapsto x\overline{a}$$

and $x\overline{a} = \overline{xa} = \overline{0}$. Hence α is injective zero map. This implies $(x) \bigotimes_A A/(x)$ is zero. And since $A/(x)$ has canonical short exact sequence

$$0 \to (x) \to A \to A/(x) \to 0,$$

tensoring with (x), which is also a flat module by i), gives us

$$0 \to (x) \bigotimes_A (x) \to A \bigotimes_A (x) = (x) \to (x) \bigotimes_A A/(x) = 0 \to 0$$

This implies $(x) \bigotimes_A (x) \cong (x)$. Since this map sends $a \otimes b \mapsto ab$, this implies that $(x) = (x)^2$. But notes that $(x)^2 = \{x^2ab : a, b \in A\} \subseteq (x^2)$ and for any $ax^2 \in (x^2)$, $ax^2 = 1 \cdot ax^2 \in (x)^2$, thus $(x) = (x^2)$. Done.

ii) $\to$ *iii)*: Since $(x) = (x)^2$ for all $x \in A$, $x = ax^2$ for some $a \in A$, thus $e = ax$ is idempotent, since $e^2 = a^2x^2 = ax = e$. Also, $(e) = (x)$ since $(e) \subseteq (x)$ is trivial, and for any $bx \in (x)$, $bx = bax^2 = (bx)(ax) = bxe$ implies $(x) \subseteq (e)$. Also, we showed in Exercise 1.11 such that $(e,f) = (e + f - ef)$ for any two idempotents e, f. Thus, every finitely generated ideal is a principal and generated by idempotent, say e. Also, $A = (e) \oplus (1-e)$ as a module, since $ae = b(1-e) \implies (a+b)e = b \implies ae + be = be \implies ae = 0 \implies a = 0$. Thus $b = 0$. And for any $x \in A$, $xe + x(1-e) = x$. Thus any finitely generated ideal (which is principal) is a direct summand of A.

iii) $\to$ *i)*. Let $\mathfrak{a}$ be a finitely generated ideal, N be an arbitrary A-module. Then, $A \cong \mathfrak{a} \oplus \mathfrak{b}$ as a module. Then,

$$N \cong N \bigotimes_A A \cong N \bigotimes \mathfrak{a} \oplus \mathfrak{b} \cong N \bigotimes_A \mathfrak{a} \oplus N \bigotimes_A \mathfrak{b}.$$

Hence, $N \bigotimes_A \mathfrak{a} \to N \bigotimes_A A$ is injective map, which is induced by canonical injection $\mathfrak{a} \to A$. Also, the short exact sequence

$$0 \to \mathfrak{a} \to A \to \mathfrak{b} \to 0$$

implies that $\mathfrak{b} \cong A/\mathfrak{a}$ as a module, thus the long exact sequence of Tor fucntor is

$$\cdots \to \mathrm{Tor}_1(\mathfrak{a}, N) \to \mathrm{Tor}_1(A, N) \to \mathrm{Tor}_1(A/\mathfrak{a}, N) \to \mathfrak{a} \otimes N \to A \otimes N \to A/\mathfrak{a} \otimes N \to 0.$$

and since A itself is free module, thus $\mathrm{Tor}_1(A, N) = 0$. Thus $\mathrm{Tor}_1(A/\mathfrak{a}, N) \cong \ker(\mathfrak{a} \otimes N \to A \otimes N) = 0$ since the given map is injective. Since $\mathfrak{a}, N$ was arbitrarily chosen, by Exercise 26, every A-module is flat, thus A is absolutely flat.

28. In a Boolean ring, every principal ideal is idempotent. Thus, it is absolutely flat. Also, if A is a ring such that $x = x^n$ for some $n \in \mathbb{N}$ (depending on x), then $x = x^n = x^2 x^{n-2}$ implies that $x \in (x^2)$, thus $(x) = (x^2)$. Hence A is absolutely flat. If B is a homomorphic image of A, then by the first isomorphism theorem, $B \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A. Then, let $(\overline{x})$ be a principal ideal of $A/\mathfrak{a}$. Since A is absolutely flat, $x = ax^2$ for some $a \in A$, thus $\overline{x} = \overline{ax}^2$. This implies $(\overline{x}) = (\overline{x}^2)$, done.

    Suppose A is a local ring and $\mathfrak{m}$ is maximal ideal. Then for any $x \in \mathfrak{m}$, $(x) = (e)$ for some idempotent e by the proof of Exercise 27, thus $e \in \mathfrak{m}$. This implies that $e = 0 \iff x = 0$. Then, $f = 1 - e$ is also idempotent, and it is unit by proposition 1.9 with the fact that in the local ring $\mathfrak{m}$ is Jacobson radical. Thus, $1 = f^{-1} f = f^{-1} f^2 = f$, hence $e = 0$, which implies $x = 0$. Thus $\mathfrak{m} = (0)$, which implies that A is a field.

    Also, if A is absolutely flat, let $x \in A$ be a nonunit. Then, by 2.27, from $(x) \neq 1$, there exists $\mathfrak{b}$ an ideal of A such that $A \cong (x) \oplus \mathfrak{b}$ as an A-module. Thus, $bx = 0$ for any $b \in \mathfrak{b}$ since $\mathfrak{b} \cap (x) = 0$. This implies that x is a zero divisor.

3 Rings and Modules of Fractions

Exercise p.37. *Verify that these definitions are independent of the choices of representatives (a, s) and (b, t), and that $S^{-1}A$ satisfies the axiom of commutative ring with identity.*

Proof. If $a/s = a'/s'$ then $\exists u \in S$ such that $(as' - a's)u = 0$. Thus

$$\begin{aligned} a/s + b/t &= (at + bs)/st = (at + bs)s'u/sts'u = ((as'u)t + bss'u)/s't(su) \\ &= ((a'su)t + bss'u)/s't(su) = (su)(a't + bs')/s't(su) = (a't + bs')/s't = a'/s' + b/t \end{aligned}$$

Also,

$$(a/s) \cdot (b/t) = ab/st = abs'u/sts'u = a'sub/suts' = a'b/s't = (a'/s') \cdot (b/t).$$

Hence multiplication and addition are well-defined. Also, it is abelian group with respect to addition (because of $0/s$) and multiplication is associative, as usual multiplication of rational numbers did, and multiplication is commutative, and s/s has a roll of identity for any $s \in S$. Thus it is commutative ring with unity. □

Proposition 3.7. *If M, N are A-modules, there is a unique isomorphism of $S^{-1}A$-modules $f : S^{-1}M \bigotimes_{S^{-1}A} S^{-1}N \to S^{-1}(M \bigotimes_A N)$ such that*

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st.$$

In particular, if $\mathfrak{p}$ is a prime ideal, then

$$M_{\mathfrak{p}} \bigotimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \cong (M \bigotimes_A N)_{\mathfrak{p}}.$$

Proof.

$$S^{-1}M \bigotimes_{S^{-1}A} S^{-1}N \underbrace{\cong}_{\text{Pro 3.5}} (S^{-1}A \bigotimes_A M) \bigotimes_{S^{-1}A} (S^{-1}A \bigotimes_A N) \underbrace{\cong}_{\text{Pro 2.14 i)}} (M \bigotimes_A S^{-1}A) \bigotimes_{S^{-1}A} (S^{-1}A \bigotimes_A N)$$
$$\underbrace{\cong}_{\text{Exe 2.15}} ((M \bigotimes_A S^{-1}A) \bigotimes_{S^{-1}A} S^{-1}A) \bigotimes_A N) \underbrace{\cong}_{\text{Exe 2.15}} (M \bigotimes_A (S^{-1}A \bigotimes_{S^{-1}A} S^{-1}A)) \bigotimes_A N$$
$$\underbrace{\cong}_{\text{2.14 iv)}} (M \bigotimes_A S^{-1}A) \bigotimes_A N \underbrace{\cong}_{\text{2.14 i)}} (S^{-1}A \bigotimes_A M) \bigotimes_A N \underbrace{\cong}_{\text{Exe 2.15}} S^{-1}A \bigotimes_A (M \bigotimes_A N)$$
$$\underbrace{\cong}_{\text{Pro 3.5}} S^{-1}(M \bigotimes_A N).$$

Also, f sends

$$\begin{aligned}(m/s) \otimes (n/t) &\mapsto (1/s \otimes m) \otimes (1/t \otimes n) \mapsto (m \otimes 1/s) \otimes (1/t \otimes n)\\ &\mapsto (m \otimes 1/st \otimes n) \mapsto 1/st \otimes (m \otimes n) \mapsto (m \otimes n)/st\end{aligned}$$

□

Proposition 3.9. *Surjectivity of an A-module homomorphism $\phi : M \to N$ is local property.*

Proof. If ϕ is surjective, then $M \to N \to 0$ is exact, thus $M_\mathfrak{p} \to N_\mathfrak{p} \to 0$ is also exact, thus $\phi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is also surjective, for any prime ideal $\mathfrak{p}$. If $\phi_\mathfrak{p}$ is surjective for any prime ideal $\mathfrak{p}$, then $\phi_\mathfrak{m}$ is also surjective for any maximal ideal $\mathfrak{m}$. Now suppose that $\phi_\mathfrak{m}$ is surjective. Let $\operatorname{coker}(\phi) = N/\operatorname{Im}(\phi) =: N'$. Then, $M \to N \to N' \to 0$ is exact. Thus, $M_\mathfrak{m} \to N_\mathfrak{m} \to N'_\mathfrak{m} \to 0$ is exact for any maximal ideal $\mathfrak{m}$. However, since $\phi_\mathfrak{m}$ is surjective, $N'_\mathfrak{m} = 0$ for any maximal ideal $\mathfrak{m}$. By Proposition 3.8, $N' = 0$. This implies that $(\phi) = N$, thus ϕ is surjective. □

Proposition 3.11. *$\mathfrak{a}$ is contracted ideal from the map $f : A \to S^{-1}A$ if and only if no elements of S is a zero divisor in $A/\mathfrak{a}$.*

Also $S^{-1}r(\mathfrak{a}) = r(S^{-1}\mathfrak{a})$.

Proof. $\mathfrak{a}$ is contracted ideal if and only if $\mathfrak{a}^{ec} \subseteq \mathfrak{a}$ by Proposition 1.17 iii). Also, $\mathfrak{a}^{ec} \subseteq \mathfrak{a}$ holds if and only if $sx \in \mathfrak{a}$ for some $s \in S$ implies $x \in \mathfrak{a}$. To see this, if $\mathfrak{a}^{ec} \subseteq \mathfrak{a}$, suppose $sx \in \mathfrak{a}$ for some $s \in S$. Then, $sx/1 \in \mathfrak{a}^e$, hence $x/1 \in \mathfrak{a}^e$, therefore $x \in f^{-1}(\mathfrak{a}^e) = \mathfrak{a}^{ec} \subseteq \mathfrak{a}$. Conversely, if the statement holds, the let $x \in \mathfrak{a}^{ec}$. Then $x/1 \in \mathfrak{a}^{ece} = \mathfrak{a}^e$ by proposition 1.17, $x/1 = a/s$ for some $a \in \mathfrak{a}, s \in S$. (This is because finite linear combination in $\mathfrak{a}^e$ can be denoted as a form a/s in $S^{-1}A$.) Hence, $sx/1 = a/1$, thus $t(sx - a) = 0$ for some $t \in S$, this implies $(st)x \in \mathfrak{a}$, and by the statement, $x \in \mathfrak{a}$.

Lastly, $sx \in \mathfrak{a}$ for some $s \in S$ implies $x \in \mathfrak{a}$ holds if and only if no $s \in S$ is a zero divisor in $A/\mathfrak{a}$. If the lefthandside holds, then let $s \in S$ such that for some $\bar{t} \in A/\mathfrak{a}$, $\overline{st} = 0$. This implies that $st \in \mathfrak{a}$. This implies $st/1 \in \mathfrak{a}^e$, hence $st/1 = a/q \implies t/1 = a/q'$ for some $a \in \mathfrak{a}, q' \in S$, thus $u(q't - a) = 0$ for some $u \in S$, hence $uq't \in \mathfrak{a}$, and from the fact $uq' \in S$ with the lefthandside implies $t \in \mathfrak{a}$, thus $\bar{t} = 0$. This shows that $\bar{s}$ is not the zero divisor in $A/\mathfrak{a}$. Conversely, suppose that no $s \in S$ is zero divisor of $A/\mathfrak{a}$, and let $sx \in \mathfrak{a}$ for some $s \in S$. Then, $\overline{sx} = 0$ implies $\bar{x} = 0$ since $\bar{s}$ is not a zero-divisor.

For the last statement, by Exercise 1.18 we have $S^{-1}r(\mathfrak{a}) \subseteq r(S^{-1}\mathfrak{a})$. Let $x/s \in r(S^{-1}\mathfrak{a})$. Then, $x^n/s^n \in S^{-1}\mathfrak{a}$. This implies that $x^n/s^n = a/t$ for some $a \in \mathfrak{a}, t \in S$. Thus, $x^n tu = s^n au$ for some $u \in S$. Hence $x^n tu \in \mathfrak{a}$, thus $xtu \in r(\mathfrak{a})$, therefore $xtu/1 \in S^{-1}r(\mathfrak{a})$, thus $xtu \cdot 1/stu = x/s \in S^{-1}r(\mathfrak{a})$, done. □

1. Let M be a finitely generated module (generated by $x_1, \cdots, x_n$ such that $S^{-1}M = 0$. Then, $x_i/1 = 0/1$ for any i, thus $\exists s_i \in S$ such that $s_i x_i = 0$. Then, $s = \prod_{i=1}^n s$ annahilates M.

   Conversely, suppose $sM = 0$ for some $s \in S$. Then, $m/t = ms/ts = 0/ts = 0/1$ for all $m/t \in S^{-1}M$, this implies $S^{-1}M = 0$.

2. Let $a/s \in S^{-1}\mathfrak{a}$ for some $a \in \mathfrak{a}, s \in S = 1 + \mathfrak{a}$. Then by Proposition 1.9, it suffices to show that for any $y/t \in S^{-1}A$, $1 - (a/s) \cdot (x/t)$ is a unit in $S^{-1}A$. To see this, notes that

$$1 - (a/s) \cdot (x/t) = 1 - ax/st = 1 - (ax)/(1 + b) \text{ for some } b \in \mathfrak{a}, \text{ hence } = \frac{1 + (b - ax)}{1 + b}$$

thus $b-ax \in \mathfrak{a}$ implies $1+(b-ax) \in S$, thus it is unit. Hence, a/s is in the Jacobson radical of $S^{-1}A$.

Now we want to show the statements that if $M = \mathfrak{a}M$ and M is finitely generated, then $\exists x \equiv 1 \mod \mathfrak{a}$ such that $xM = 0$.

suppose that $M = \mathfrak{a}M$. Then, $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$ by Proposition 3.11 v). By the above proof, we know that $(S^{-1}\mathfrak{a})$ is in the Jacobson radical of $S^{-1}A$, and $S^{-1}M$ is still finitely generated; (if M is generated by x_is, then $S^{-1}M$ is generated by $x_i/1$s.) Thus by the Proposition 2.6 Nakayama's lemma, $S^{-1}M = 0$. By above exercise 3.1, $\exists x \in S$ such that $xM = 0$. Since $S = 1+\mathfrak{a}$, $x = 1+a$ for some $a \in \mathfrak{a}$, done.

3. Let $f : (ST)^{-1}A \to U^{-1}(S^{-1}A)$ by $x/st \mapsto (x/s)/(t/1)$. First of all, it is well-defined; if $x/st = x'/s't'$, then $xs't's_1t_1 = x'sts_1t_1$ for some $q \in ST$.

$$(x'/s')/(t'/1) = (x'sts_1t_1/s'ss_1)/(t_1tt'/1) = (xs't's_1t_1/s'ss_1)/(t_1tt'/1) = (xt't_1/s)/(t't_1t/1) = (x/s)/(t/1).$$

where last equality comes from the observation that $(x/s)/(t/1) = (x/s)/(t/1) \cdot 1/1 = (x/s)/(t/1) \cdot (t't_1/1)/(t't_1/1) = (xt't_1/s)/(t't_1t/1)$.

Also, if f mapst x/st to 0, then $(x/s)/(t/1) = (0/s)/(t/1)$, thus $(t'/1)(t/1)(x/s-0/s) = 0$ for some $t' \in T$, thus $tt'x/s-0/s) = 0$, which implies $tt'sx = 0$. This implies that $(x/st) = xtt's/sttt's = 0/s^2t^2t' = 0$. So f is injective. And, for any element in $U^{-1}(S^{-1}A)$ can be denoted as $(x/s)/(t/1)$, thus it has preimage x/st, which shows that f is surjective. And f sends $1/1 \cdot 1$ to $(1/1)/(1/1)$, so it sends 1 to 1, and

$$f\,(x/st + y/s't') = f((xs't'+yst)/ss'tt') = (xs't'+yst)/ss'/(tt'/1) = ((xt'/s)+(yt/s'))/(tt'/1) = (x/s)/(t/1) + (y/s'$$

$$f\,(x/st \cdot y/s't') = (xy/ss')/(tt'/1) = ((x/s)\cdot(y/s'))/((t/1)\cdot(t'/1)) = f(x/st)\cdot f(y/s't').$$

4. Notes that B is an A-algebra along $a.b := f(a)b$. Thus $S^{-1}B = \{x/s : x \in B, s \in S\}$ is an A-module. Also, $T^{-1}B = \{x/f(s) : x \in B, f(s) \in T\}$. Hence, take $\phi : S^{-1}B \to T^{-1}B$ as $\phi(x/s) = x/f(s)$. First of all, it is well-defined, since for any $x/s = x'/s'$, $f(t)(xf(s')-x'f(s)) = 0$ for some $t \in S$, thus $x/f(s) = x'/f(s')$. Also, it is injective since $\phi(x/s) = 0$ implies $f(t')f(s)x = 0$, thus $x/s = 0/s$ since this is equivalent to say there exists $t \in S$ such that $xf(s)f(t) = 0$. Also, it is surjective, and it is clearly homomorphism.

5. By Corollary 3.12 and the given condition, $\mathfrak{N}_\mathfrak{p} = 0$ for all prime ideal $\mathfrak{p}$. By Proposition 3.8, this implies $\mathfrak{N} = 0$.

The answer for next question is no. Suppose that $A = \prod_{i=1}^n k_i$, where $k_i = k$ an algebraic closed field. This is not integral domain, since $(1,0,\cdots,0)\cdot(0,1,0,\cdots,0) = 0$. Then, by Execise 1.22, $\mathrm{Spec}(A) = \{\mathfrak{p}_i := 0 \times \prod_{j\neq i}^n k_j\}$. Thus $A - \mathfrak{p}_i = k_j^\times \prod_{j\neq i}^n K_j$. Then, also notes that each k_i as embedded in A is an ideal, thus an A-module, this implies $A = \oplus_{j=1}^n k_j$. Hence, if we let $S = A-\mathfrak{p}_i$, then

$$S^{-1}k_j = \begin{cases} k_j & \text{if } i = j \\ 0 & \text{o.w.}. \end{cases}$$

To see this, if $i \neq j$, then $e_i \cdot k_j = 0$ since we regard k_j be an elements in A whose tuple expression has only nonzero component on j-th position. Since $e_i \in S$, by Exercise 3.1, $S^{-1}k_j = 0$. If $i = j$, then its elements has a form x/s where $x \in k_i$, s is a tuple in A such that i-th components is nonzero. Now define a map $f : S^{-1}k_j \to k_j$ by $(0,\cdots,0,x_i,0,\cdots,0)/(s_1,\cdots,s_i,\cdots,s_n) \mapsto x_i/s_i$. First of all, it is well-defined; if $x/s = x'/s'$, then there exists $t \in S$ such that $t(xs'-x's) = 0$. This implies that $t_ix_is_i' = t_ix_i's_i$, and since all variables in the equation is from the same field, so we can get rid of t_i, thus $x_is_i' = x_i's_i$, this implies $x_i/s_i = x_i'/s_i'$ since s_i, s_i' are nonzero. Also, it sends 1 to 1, and it is surjective, since for any $x_j \in k_j$, $f(0,\cdots,0,x_j,0,\cdots,0)/(1,\cdots,1) = x_j$. Also it is injective, since $f(x/s) = 0$ implies that $x_i/s_i = 0$ implies $x_i = 0$. Hence, using Proposition 3.11 saying that S^{-1} commutes with finite sums,

$$S^{-1}A = \bigoplus_{j=1}^n S^{-1}k_j \cong k_j.$$

Hence for any prime ideal $\mathfrak{p}_i$, the localization at $\mathfrak{p}_i$ is integral domain. But A itself is not integral domain.

6. First of all $\{1\} \in \Sigma$ hence it is nonzero. And if C is a chain in Σ, then $U = \bigcup_{c \in C} c$ is also a multiplicatively closed sets. To see this, if $x, y \in U$, then $\exists c_x, c_y \in C$ such that $x \in c_x, y \in c_y$, and since C is chain, one of c_x or c_y is contained in the other. WLOG, suppose $c_x \supseteq c_y$. Then, $xy \in C_x \subseteq U$. Also U is maximal elements of C. Hence, by the Zorn's lemma, Σ has a maximal element.

   Now suppose $S \in \Sigma$. We claim that $A - S$ has a prime property, i.e., $xy \in A - S$ implies $x \in A - S$ or $y \in A - S$. To see this, suppose not. Then, both $x, y \in S$, thus $xy \in S$, contradiction. Also, we see that $A - S$ is an ideal when S is maximal multiplicative subset. To see this, notes that for any $a \in A$, the smallest multiplicative set containing a and S is S when $a \in S$, and $S_a = \{sa^n : s \in S, n \in \mathbb{N}\}$ when $a \in A - S$. However, since S is maximal in Σ, S_a strictly contains S implies $0 \in S_a$. Thus, $sa^n = 0$ for some $n > 0$, since $0 \notin S$. Thus, suppose that $a, b \in A - S$. Then, $sa^n = 0, tb^m = 0$ for some $s, t \in S, n, m \in \mathbb{N}$. This implies that for $p = n + m - 1$, $a^n | (a + b)^p, b^m | (a + b)^p$, hence $st(a + b)^p = 0$. Thus, $a + b \in A - S$. Also, for any $x \in A$, $a \in A - S$, $s(ax)^n = (sa^n)x^n = 0x^n = 0$ implies $ax \in A - S$. Hence $A - S$ is a prime ideal. Also, it is minimal since otherwise, if $\mathfrak{p} \subseteq A - S$, this implies $T := A - \mathfrak{p} \supseteq S$ and $0 \notin T$, hence $T \in \Sigma$, which implies $T = S$ by maximality of S, thus $A - \mathfrak{p} = S$, thus $\mathfrak{p} = A - S$.

   Conversely, suppose $A - S$ is a minimal prime ideal of A. Then, there exists $T \in \Sigma$ which is a maximal elements and $S \subseteq T$, thus $A - T$ is prime ideal contained in $\mathfrak{p}$. However, by minimality of $A - S$, $A - S = A - T$, which implies $S = T$, thus S is maximal element in Σ. Conversely, if S is a maximal elements, then $A - S$ is a prime ideal, and if there exists a prime ideal $\mathfrak{p}$ containing $A - S$, let $T = A - \mathfrak{p}$, and since $0 \in \mathfrak{p}$, $T \in \Sigma$, and $T \subseteq S$. Maximality of S implies $T = S$, thus $\mathfrak{p} = A - S$. Hence S is a maximal element in Σ.

7. (a) Suppose $A - S = \bigcup_{i \in I} \mathfrak{p}_i$ where $\mathfrak{p}_i$ is a prime ideal in A. Since $x, y \in S \implies xy \in S$ is clear by definition of S, we should verify $xy \in S$ implies $x, y \in S$. Suppose $x \in A - S$. Then for any $y \in A$, since $x \in \mathfrak{p}_i$ for some $i \in I$, $xy \in \mathfrak{p}_i \subseteq A - S$. Thus its contrapositive says the desired result.

   Conversely, suppose that S is saturated. It suffices to show that for any $a \in A - S$, a prime ideal containing a is disjoint with S. Now let Σ be a collection of ideal containing a and disjoint with S. First of all, $(a) \cap S = 0$ since $xa \in S$ implies $a \in S$, contradiction. Hence $(a) \in \Sigma$. And for any chain of ideals in Σ, its union is a maximal elements of the chain by the usual argument. Hence this Σ has a maximal elements, say $\mathfrak{p}$. Then, for any maximal elements $\mathfrak{p}$ in Σ, we claim that $\mathfrak{p}$ is a prime. To see this, if $x, y \notin \mathfrak{p}$ then $(x) + \mathfrak{p}$ and $(y) + \mathfrak{p}$ contains $\mathfrak{p}$ strictly, but not in Σ, thus $(x) + \mathfrak{p} \cap S \neq \emptyset \neq (y) + \mathfrak{p}$. Let $s \in (x) + \mathfrak{p} \cap S, t \in (y) + \mathfrak{p} \cap S$. Then, $st \in ((x) + \mathfrak{p})((y) + \mathfrak{p}) \subseteq (xy) + \mathfrak{p}$, hence $xy \notin \mathfrak{p}$.

   (b) Define $\overline{S}$ be a complement of the union of the prime ideals which do not meet S. Then, definitely, $\overline{S} \supseteq S$. Also, if there exists another saturated multiplicative monoid T containing S and subset of $\overline{S}$, then, $A - T$ is union of prime ideals, and each of this ideal do not meet T, thus do not meet S, hence it is contained in $A - \overline{S}$. This implies $A - T \subseteq A - \overline{S}$ implies $T \supseteq \overline{S}$, hence $T = \overline{S}$. Thus, $\overline{S}$ is the unique smallest saturated multiplicatively closed subsets containing S.

   If $S = 1 + \mathfrak{a}$, then a prime ideal $\mathfrak{p}$ meets S if $x = 1 + a \in \mathfrak{p}$ for some $a \in \mathfrak{a}$. This implies $1 = x - a$, hence $\mathfrak{a} + \mathfrak{p} = (1)$, i.e., coprime. Conversely, if $\mathfrak{a} + \mathfrak{p} = (1)$, then $a + x = 1$ implies $x = 1 - a$, thus $S \cap \mathfrak{p} \neq \emptyset$. This implies that $\overline{S}$ is a complement of the union of all prime ideals which is not coprime with $\mathfrak{a}$. Notes that for each prime ideal $\mathfrak{p}$ not coprime with $\mathfrak{a}$, $\mathfrak{p} + \mathfrak{a}$ is contained in a maximal ideal $\mathfrak{m}$, since they are not coprime. Also, definitely, this $\mathfrak{m}$ is contained in the union. Thus,

$$\overline{S} = A \setminus \bigcup_{\substack{\mathfrak{m} \in \operatorname{Spec} M(A) \\ \mathfrak{m} \supseteq \mathfrak{a}}} \mathfrak{m}$$

   where $\operatorname{Spec} M(A)$ is a set of all maximal ideals of A.

8. $i) \to ii)$: For given $t \in T$, $\phi^{-1}(1/t)$ exists in $S^{-1}A$ by bijectivity of ϕ. Hence, $\phi(t/1 \cdot \phi^{-1}(1/t)) = \phi(t/1) \cdot 1/t = t/1 \cdot 1/t = 1/1$ and $\phi(1) = 1$ by bijectivity implies that $t/1 \cdot \phi^{-1}(1/t) = 1/1$. Hence, $t/1$ is a unit.

ii) $\to$ *iii*): Since $t/1$ is a unit in $S^{-1}A$, there exists $a/s \in S^{-1}A$ such that $at/s = 1/1$. Hence, $q(at - s) = 0$ for some $q \in S$, thus $qat = qs \in S$. By letting $x = qa$, done.

iii $\to$ *iv*): If $t \in T$, by iii), $\exists a \in A$ such that $at \in S \subseteq \overline{S}$, this implies

$$a \in \overline{S}, t \in \overline{S}.$$

Hence $T \subseteq \overline{S}$.

iv) $\to$ *v*): If a prime ideal $\mathfrak{p}$ do not meet S, then it doesn't meet $\overline{S}$ by definition, thus it doesn't meet T. Its contrapositive is what we want to have.

v) $\to$ *i*): To see the injectivity, we claim an existence of prime ideal with respect to some ideal outside of S.

Claim XIII. *Let S be a multiplicative subset, and $\mathfrak{a}$ be an ideal such that $\mathfrak{a} \cap S = \emptyset$. Then there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap S = \emptyset$ and $\mathfrak{p} \supseteq \mathfrak{a}$.*

Proof. Standard Zorn's lemma proof. Let Σ be an ideal containing $\mathfrak{a}$ and contained in $A - S$ ordered by inclusion. Then $\mathfrak{a} \in \Sigma$ and any chain has a a maximal element, which is union of all elements in the chain. Thus, by Zorn's lemma $\exists \mathfrak{p} \in \Sigma$ a maximal element.

Then, $\mathfrak{p}$ is an ideal, and $A - \mathfrak{p}$ is multiplicative set containing S. Thus, if $x, y \notin \mathfrak{p}$, then $(x) + \mathfrak{p}, (y) + \mathfrak{p}$ are ideals meeting S by maximality. Hence, take $s \in ((x) + \mathfrak{p}) \cap S, t \in ((y) + \mathfrak{p}) \cap S$ then $st \in ((xy) + \mathfrak{p}) \cap S$. This implies $xy \notin \mathfrak{p}$, thus $\mathfrak{p}$ is prime ideal. □

This induces another claim.

Claim XIV. *Let $\mathfrak{a}$ be an ideal of A. If $f \notin r(\mathfrak{a})$, then $\exists \mathfrak{p}$ a prime ideal containing $\mathfrak{a}$ such that $f \notin \mathfrak{p}$.*

Proof. Let $S = \{1, f, f^2, \cdots\}$ and use above claim to get $\mathfrak{p}$. Then, $\mathfrak{p} = r(\mathfrak{p}) \supseteq r(\mathfrak{a})$. □

Let $\phi(a/s) = 0/1$ in $T^{-1}A$. Then $\exists t \in T$ such that $ta = 0$. It suffices to show that $a/s = 0/1$ in $S^{-1}A$. If not, then $\mathrm{Ann}(a) \cap S = \emptyset$. Hence, $\mathrm{Ann}(a) \subseteq A - S$. By the above claim $\exists \mathfrak{p}$ a prime ideal containing $\mathrm{Ann}(a)$ and $\mathfrak{p} \cap S = \emptyset$. This implies $\mathfrak{p}$ doesn't meet T by v), so $t \notin \mathrm{Ann}(a)$, contradiction. Hence $a/s = 0/1$ in $S^{-1}A$. So ϕ is injective.

To see surjectivity, we claim that $\forall t \in T$, $\exists a \in A$ such that $at \in S$. Otherwise, $(t) \cap S = \emptyset$ for any $t \in T$ thus $AT \cap S = \emptyset$, thus by the claim, there is a prime ideal $\mathfrak{p}$ containing AT, and $\mathfrak{p} \cap S = \emptyset$. This implies $\mathfrak{p} \cap T = \emptyset$ by v), contradiction since $\mathfrak{p} \supseteq AT \supseteq T$. Thus the claim is true, and by the claim, for any b/t. $b/t = ab/at$ thus $\phi(ab/at) = ab/at = b/t$.

9. S_0 is saturated is clear; just pick product and assume one of productee is zero divisor. Now we claim that S_0 is contained in any maximal elements in Σ from Exercise 3.6. This shows that $D := A - S_0$ contains $A - S$, a minimal prime ideal. And by Exercise 3.6, every minimal prime ideal corresponds to maximal elements in Σ, this shows that D contains every minimal prime ideal.

   To see S_0 is contained in any $S \in \Sigma$ which is a maximal in Σ, suppose not; then S_0S is a multiplicatively closed subsets containing S strictly. Thus S_0S should contain 0 since it is not in Σ. Thus $s_0s = 0$ for some $s_0 \in S_0, s \in S$, which implies that s_0 is a zero-divisor, contradiction.

   (a) If S contains S_0 strictly, then S contains a zero divisor, say $b \in S$ and $bc = 0$ for some $c \in A$. Hence, for $f : A \to S^{-1}A$ by $a \mapsto a/1$, then $f(c) = c/1 = cb/b = 0/c = 0$. Thus $c \in \ker f \neq 0$, so f is not injective. Also we should show that $f : A \to S_0^{-1}A$ is injective. To see this, if $f(a) = a/1 = 0$, then $sa = 0$ for some $s \in S_0$, but since s is non-zero-divisor, $a = 0$.

   (b) Let $a/s \in S_0^{-1}A$ for some $a \in A, s \in S_0$. Then, if $a \in S_0$, then a/s is unit; since $s/a \cdot a/s = 1/1$. If $a \notin S_0$, then a is zero-divisor by definition of S_0, thus $b \in A$ such that $ab = 0$, hence $a/s \cdot b/s = ab/s^2 = 0/s^2 = 0/1$.

(c) Let $\phi : A \to S_0^{-1}A$. Then, $\phi(a) = 0$ implies $a/1 = 0/1$ thus $as = 0$ for some $s \in S_0$. Since S_0 consists of all units, $as = 0$ implies $a = 0$. Thus ϕ is injective. Also, surjectivity is clear; for any $a/s \in S_0^{-1}A$, $as^{-1}/1 = a/s$, thus $\phi(as^{-1}) = as^{-1}/1 = a/s$, done.

10. (a)

Claim XV. *X Let M be a $S^{-1}A$ module. Then, it can be regarded as A-module in a canonical way using the canonical map $A \to S^{-1}A$. Then,*

$$M \cong S^{-1}M|_A \text{ as a } S^{-1}A\text{-module}$$

where $M|_A$ is the same M as an A-module.

Proof. Let $\phi : M \to M|_A \to S^{-1}M|_A$ be a map sending m to $m/1$. First of all, ϕ is injective since $\phi(m) = 0$ implies $m/1 = 0/1$ implies $\exists s \in S$ such that $sm = 0$ in $M|_A$. Hence, $(s/1) \cdot m = 0$. Hence $(1/s) \cdot (s/1) \cdot m = 0$ implies $m = 0$. Also, it is $S^{-1}A - homomorphism$, since for any $a/s \in S^{-1}A$ and $m \in M$

$$\phi((a/s) \cdot m) = (am/s)/1 = s(am/s)/s = (am/1)/s = (a/s) \cdot (m/1),$$

and additivity holds trivially. Now, ϕ is also surjective homomorphism since for any $m/s \in S^{-1}(M|_A)$, $\phi((1/s) \cdot m) = 1/s \cdot (m/1) = m/s$. ∎

Now suppose that A is absolutely flat. Take M be any $S^{-1}A$ module. Then $M|_A$ is also flat, hence, for any injective $S^{-1}A$-module injective map $f : N \to N'$,

$$\mathfrak{M} \bigotimes_A f : N|_A \bigotimes_A S^{-1}M|_A \to N'|_A \bigotimes_A M|_A$$

is injective as an A-module homomorphism. Then by Proposition 3.3,

$$S^{-1}(N|_A \bigotimes_A S^{-1}M|_A) \to S^{-1}(N'|_A \bigotimes_A S^{-1}M|_A)$$

is injective as a $S^{-1}A$-module homomorphism. Now By Proposition 3.7,

$$S^{-1}(N|_A \bigotimes_A S^{-1}M|_A) \cong S^{-1}N|_A \bigotimes_{S^{-1}A} S^{-1}M|_A \text{ and } S^{-1}(N'|_A \bigotimes_A S^{-1}M|_A) \cong S^{-1}N'|_A \bigotimes_{S^{-1}A} S^{-1}M|_A$$

as a $S^{-1}A$-module. Thus, a map

$$S^{-1}N|_A \bigotimes_{S^{-1}A} S^{-1}M|_A \to S^{-1}(N|_A \bigotimes_A S^{-1}M|_A) \to S^{-1}(N'|_A \bigotimes_A S^{-1}M|_A) \to S^{-1}N'|_A \bigotimes_{S^{-1}A} S^{-1}M|_A$$

is injective as a $S^{-1}A$-module homomorphism. Since $S^{-1}N|_A \cong N$, $S^{-1}AS^{-1}M|_A \cong M$, and $S^{-1}AS^{-1}N'|_A \cong N'$, this implies that $N \bigotimes_{S^{-1}A} M \to N' \bigotimes_{S^{-1}A} M$ is injective. Thus M is flat. Since we take an arbitrary $S^{-1}A$-module M, $S^{-1}A$ is also absolutely flat.

(b) If A is absolutely flat, then $A_{\mathfrak{m}}$ is absolutely flat by i). By Exercise 2.28 stating that absolutely flat local ring is a field, $A_{\mathfrak{m}}$ is a field. Conversely, suppose that $A_{\mathfrak{m}}$ is a field for any maximal ideal $\mathfrak{m}$. Let M be an A-module. Then, $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-module, and since $A_{\mathfrak{m}}$ is a field, $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module. (Since every module over a field is vector space, and vector space always has a basis.) And every free module is flat. Hence $M_{\mathfrak{m}}$ is flat for all $\mathfrak{m}$. By Proposition 3.10, M is flat. Since M was arbitrarily chosen, A is absolutely flat.

11. *i)* $\to$ *ii)*. Suppose that there is a prime ideal $\mathfrak{p}$ strictly contained in a maximal ideal $\mathfrak{m}$. Then, they are still prime and maximal even if we send it to $A/\mathfrak{R}$, since by Proposition 1.1 this correspondence of ideal is order preserving, and by Exercise 1.21 iv), $\mathrm{Spec}(A/\mathfrak{R}) \to \mathrm{Spec}(A)$ is natural homeomorphism. Denote $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{m}}$ are image of $\mathfrak{p}$ and $\mathfrak{m}$ in $A/\mathfrak{R}$. Hence, this implies that $\overline{\mathfrak{p}}_{\overline{\mathfrak{m}}}$ and $\overline{\mathfrak{m}}_{\overline{\mathfrak{m}}}$ are prime and

maximal ideals in $(A/\mathfrak{R})_{\overline{\mathfrak{m}}}$ by Corollary 3.13. Notes that Corollary 3.13 also implies that image of $\mathfrak{p}$ strictly lies in that of $\mathfrak{m}$. By Exercise 3.10, $(A/\mathfrak{R})_{\overline{\mathfrak{m}}}$ is a field, thus $(0) = \overline{\mathfrak{m}}_{\overline{\mathfrak{m}}}$,by maximality. This implies $\overline{\mathfrak{m}}_{\overline{\mathfrak{m}}} = \overline{\mathfrak{p}}_{\overline{\mathfrak{m}}}$, contradiction.

$ii) \to iv)$: Let $\mathfrak{p}_x \neq \mathfrak{p}_y \in \mathrm{Spec}(A/\mathfrak{R})$. Then, $\mathfrak{p}_x$ and $\mathfrak{p}_y$ are distinct maximal ideals by ii). Thus $\mathfrak{p}_x + \mathfrak{p}_y$ is unit ideal, since it is an ideal strictly containing both maximal ideal. Thus, $\exists a \in \mathfrak{p}_x, b \in \mathfrak{p}_y$ such that $a + b = 1$. This implies $(a) + (b) = 1$. By Exercise 2.27 ii), $(a) = (e), (b) = (g)$ for some idempotents e, g. Thus $(e, g) = 1$. Hence, let $f = e(1 - g)$. Then, $gf = e(g - g^2) = 0$. But $e = eg + f \in (g, f)$. Hence, $(g, f) = (1)$. Since $g \in (b) \subseteq \mathfrak{p}_y$, we have $y \in X_g$. Similarly, since $f \in \mathfrak{p}_x$, $x \in X_f$. Now, $X_f \cap X_g = X_{fg} = X_0 = \emptyset$ by Exercise 1.17. i), ii). Hence, X_f and X_g are two disjoint open sets in $\mathrm{Spec}(A/\mathfrak{R})$ separating x and y. Hence $\mathrm{Spec}(A/\mathfrak{R})$ is Hausdorff, so does $\mathrm{Spec}(A)$ by natural homeomorphism.

$iv) \to iii)$: Hausdorff is $T2$ space, so it is $T1$.

$iii) \to ii)$: If every singleton is closed, this implies that every prime ideal is maximal by Exercise 1.18.

$ii) \to i)$: Suppose that every prime ideal of A is maximal. Then, by order preserving 1-1 correspondence from Proposition 1.1, every prime ideal of $A/\mathfrak{R}$ is maximal. Thus, for any maximal ideal $\mathfrak{m}$ of $A/\mathfrak{R}$, $\mathfrak{m}_{\mathfrak{m}}$ is the only maximal ideal of $(A/\mathfrak{R})_{\mathfrak{m}}$ and also the only minimal prime ideal of $\mathfrak{m}_{\mathfrak{m}}$, thus $\mathfrak{m}_{\mathfrak{m}}$ is nilradical in $(A/\mathfrak{R})_{\mathfrak{m}}$. However, $A/\mathfrak{R}$ is integral domain, thus there is no nilpotent elements, which implies $\mathfrak{m}_{\mathfrak{m}} = (0)$. Hence $(A/\mathfrak{R})_{\mathfrak{m}}$ is field. Since $\mathfrak{m}$ was arbitrarily chosen, this implies that $A/\mathfrak{R}$is absolutely flat by Exercise 3.10.

12. We want to show that $T(M)$ is a submodule of M. Notes that for any torsion elements $x, y \in T(M)$, $\exists a, b \in A$ such that $ax = 0, by = 0$ Hence $ab(x + y) = 0$ implies $x + y \in T(M)$. Also, for any $a \in A$, $ax \in T(M)$ since $b \in \mathrm{Ann}(x)$ also annihilates ax. Thus, $T(M)$ is submodule of M.

    (a) Let $\overline{x} \in M/T(M)$ such that $\overline{x} \neq \overline{0}$. If $a \in \mathrm{Ann}(\overline{x})$, then $a.\overline{x} = \overline{ax} = 0$ implies $ax \in T(M)$. Thus $\exists b \in A$ such that $bax = 0$ and $b \neq 0$. However, since $x \notin T(M)$, (otherwise $\overline{x} = \overline{0}$), $ba = 0$. Since A is integral domain, $a = 0$. Thus $\mathrm{Ann}(\overline{x}) = 0$. Hence $M/T(M)$ is torsion free.
    (b) Let $x \in T(M)$ and $f(x) \in N$. Then, take nonzero $b \in \mathrm{Ann}(x)$. Thus $bf(x) = f(bx) = f(0) = 0$ implies $f(x) \in T(N)$.
    (c) To see that torsion functor is left exact, By ii), if $M' \to M$ is injective, then $T(M') \subseteq T(M)$ is also injective. Also, if we let $f : M' \to M$ and $g : M \to M''$, then $f|_{T(M')}$ is injective as we've shown and $f|_{T(M')} \circ g|_{T(M)} = 0$ implies that $\mathrm{Im}\, f|_{T(M')} \subseteq \mathrm{Im}\, g|_{T(M)}$. To see the other direction, let $x \in \ker g|_{T(M)}$. Then, by exactness of original sequence, $\exists y \in M'$ such that $f(y) = g$. Now we claim that $y \in T(M')$. To see this, since $x \in T(M)$, $\exists a \in A$ such that $a \neq 0$ and $ax = 0$. This implies $af(y) = 0$. Hence, $f(ay) = 0$ By injectivity of f, $ay = 0$ Hence $y \in T(M')$. Thus, $\ker g|_{T(M)} \subseteq \mathrm{Im}\, f|_{T(M')}$, done.
    (d) If we let $S = A - \{0\}$, then $K \bigotimes_A M \cong S^{-1}M$ by Proposition 3.5. with exact map $a/b \otimes m \mapsto am/b$. Thus, $1 \otimes m = 0$ if and only if $m/1$ is zero in $S^{-1}M$ if and only if $\exists a \in S$ such that $am = 0$ by Exercise 3.1.

13. Notes that both $T(S^{-1}M)$ and $S^{-1}(TM)$ are submodule of $S^{-1}M$. Now let $m/s \in T(S^{-1}M)$. Then, $\exists a \in A$ such that $a \neq 0$ and $a \cdot m/s = am/s = 0$. This implies that $\exists t \in S$ such that $tam = 0$ in M. Since $ta \neq 0$, $m \in T(M)$. Hence $m/s \in S^{-}(TM)$. Conversely, let $m/s \in S^{-1}(TM)$. Then, $\exists a \in A$ such that $a \neq 0$ and $am = 0$ in A. Hence, $m/s \in T(S^{-1}M)$ since $a \cdot m/s = am/s = 0/s$.

    To see the TFAE,

    $i) \to ii)$: $T(M_{\mathfrak{p}}) = (T(M))_{\mathfrak{p}} = 0_{\mathfrak{p}} = 0$.

    $ii) \to iii)$: clear.

    $iii) \to i)$: Let $l_a : M \to M$ by $x \mapsto ax$. It suffices to show that l_a is injective for all $a \in A \setminus \{0\}$. Notes that by given condition, $(l_a)_{\mathfrak{m}} : M_{\mathfrak{m}} \to M_{\mathfrak{m}}$ by $x/s \mapsto ax/s$ is injective for all $a \in M$ and for all maximal ideal $\mathfrak{m}$. To see this, suppose that there exists l_a which is not injective for some $a \in A$ and some maximal ideal $\mathfrak{m}$. Then, $\exists m/s \in M_{\mathfrak{m}}$ such that $ax/s = 0$. Since $M_{\mathfrak{m}}$ is torsion free,

$a.(x/s) = ax/s = 0$ implies $a = 0$, contradiction. Hence, by Proposition 3.19, l_a is injective for all $a \in A \setminus \{0\}$. This implies that M is torsion free.

14. By Proposition 1.1, every maximal ideal in $A/\mathfrak{a}$ corresponds to every maximal ideal in A containing $\mathfrak{a}$. Now,

$$0 = M_{\mathfrak{m}}/(\mathfrak{a}M)_{\mathfrak{m}} \underbrace{\cong}_{\text{Cor 3.4 iii)}} (M/\mathfrak{a}M)_{\mathfrak{m}} \underbrace{\cong}_{\text{Prop 3.5}} A_{\mathfrak{m}} \bigotimes_A M/\mathfrak{a}M.$$

as $A_{\mathfrak{m}}$-module. Now for any $a/s \in \mathfrak{a}_{\mathfrak{m}}$,

$$a/s \otimes \overline{m} = 1/s \otimes a\overline{m} = 0.$$

This implies that we can regard $A_{\mathfrak{m}} \bigotimes_A M/\mathfrak{a}M$ as $A/\mathfrak{a}$-module isomorphic to $(A/\mathfrak{a})_{\mathfrak{m}} \bigotimes_{A/\mathfrak{a}} M/\mathfrak{a}M$. This is because all $\mathfrak{a}_{\mathfrak{m}}$ part in $A_{\mathfrak{m}}$ is zero, and all action over $\mathfrak{a}$ is also zero. Hence,

$$(A/\mathfrak{a})_{\mathfrak{m}} \bigotimes_{A/\mathfrak{a}} M/\mathfrak{a}M \qquad \underbrace{\cong}_{\text{Prop 3.5 and some fact}} \qquad (M/\mathfrak{a}M)_{\mathfrak{m}/\mathfrak{a}} = 0$$

for any maximal ideal in $A/\mathfrak{a}$. By proposition 3.8, $M/\mathfrak{a}M$ is 0 as $A/\mathfrak{a}$-module. This implies that $M = \mathfrak{a}M$.

Now we will prove the “some fact.”

Claim XVI. *XI* $(A/\mathfrak{a})_{\mathfrak{m}} \cong (A/\mathfrak{a})_{\mathfrak{m}/\mathfrak{a}}$ *as* $A/\mathfrak{a}$*-module.*

Proof. Let $\phi : (A/\mathfrak{a})_{\mathfrak{m}} \to (A/\mathfrak{a})_{\mathfrak{m}/\mathfrak{a}}$ by $\overline{a}/s \mapsto \overline{a}/\overline{s}$. First of all, it is well-defined; if $\overline{a}/s = \overline{a'}/s'$, then $\exists t \in A - \mathfrak{m}$ such that $t\overline{a}s' = t\overline{a'}s$, which is equivalent to say that $\overline{tas'} = \overline{ta's}$. This is equivalent to say that $\overline{a}/\overline{s} = \overline{a'}/\overline{s'}$.

Also, it is additive homomorphism since

$$\phi(\overline{a}/s + \overline{b}/t) = \overline{at + bs}/\overline{st} = \overline{a}/\overline{s} + \overline{a}/\overline{t}.$$

Also, for any $a \in A$,

$$a.\phi(\overline{b}/s) = a.(\overline{b}/\overline{s}) = \overline{ab}/\overline{s} = \phi(\overline{ab}/s)$$

This implies that ϕ is A-module homomorphism. Also, we already know that for any $\overline{b}/s \in (A/\mathfrak{a})_{\mathfrak{m}}$, and $a \in \mathfrak{a}$,

$$a.(\overline{b}/s) = \overline{ab}/s = 0/s.$$

Hence any two representative a and a' of $\overline{a}, a = a' + t$ for some $t \in \mathfrak{a}$, thus

$$a.(\overline{b}/s) = \overline{ab}/s = \overline{a'b}/s = a'.\overline{b}/s.$$

This implies that $(A/\mathfrak{a})_{\mathfrak{m}}$ has a natural $A/\mathfrak{a}$-module structure. Hence, for any $\overline{c} \in A/\mathfrak{a}$,

$$\phi(\overline{c}.\overline{b}/s) = \overline{bc}/\overline{s} = \overline{c}.(\overline{b}/\overline{s}) = \overline{c}\phi(\overline{b}/s).$$

Thus ϕ is $A/\mathfrak{a}$-module homomorphism. Finally, surjectivity is clear; just take representatives. Injectivity comes from the observation that

$$\phi(\overline{b}/s) = \overline{b}/\overline{s} = 0/\overline{1} \implies \overline{bs} = 0 \implies bs \in \mathfrak{a}.$$

Then, $\overline{b}/s = \overline{bs}/s^2 = 0/s^2$, thus done. □

15. Let $x_1, \cdots, x_n$ be a set of generators and $e_1, \cdots, e_n$ be the canonical basis of F. Then, let $\phi : F \to F$ by $\phi(e_i) = x_i$. Then, ϕ is surjective since any element can be denoted as $\sum_{j=1}^n a_j x_j$, which is image of $\sum_{j=1}^n a_i e_i$. To show ϕ is isomorphism, need to show that ϕ is injective. By Proposition 3.9, it suffices to show that $\phi_{\mathfrak{m}}$ is injective for any maximal ideal $\mathfrak{m}$. Assume that A is local ring. (Or, thinking

about the case $A_{\mathfrak{m}}$ for a fixed maximal ideal $\mathfrak{m}$, and working with maximal ideal $\mathfrak{m}A_{\mathfrak{m}}$.) Fix $\mathfrak{m}$ and let $N = \ker\phi$ and let $k = A/\mathfrak{m}$ be the residue field of A. Now notes that Tor sequence over k gives an exact sequence

$$\cdots \to \mathrm{Tor}_1^A(k, F) \to k \bigotimes_A N \to k \bigotimes_A F \to k \bigotimes_A F \to 0.$$

Since F is flat, $\mathrm{Tor}_1^A(k, F) = 0$ by Exercise 2.24 (3) and balancing Tor theorem. Thus,

$$0 \to k \bigotimes_A N \to k \bigotimes_A F \to k \bigotimes_A F \to 0.$$

is exact. And $k \bigotimes_A F = k \bigotimes_A \bigoplus_{j=1}^n A = \bigoplus_{j=1}^n (k \bigotimes_A A) = k^n$. Thus, k^n has a k-vector space structure. And given map $1 \otimes \phi : k \bigotimes_A \to F \to k \bigotimes_A F$ induces a map $\phi' : k^n \to k^n$ by

$$q \otimes x_i \mapsto q \otimes \phi(x_i) = q \otimes e_i \mapsto qe_i.$$

Also, notes that $\{1 \otimes x_i\}$ forms a basis of k^n since every basic elements in $k \bigotimes_A F$ is a sum of the forms $\overline{q} \otimes a_i x_i = \overline{a_i q} \otimes x_i$, and $\dim k^n = n$ implies $\{1 \otimes x_i\}$ is linearly independent. By similar argument, $\{1 \otimes e_i\}$ forms a basis of k^n. Also, $1 \otimes \phi$ maps a basis to another basis, and it is linear by definition of ϕ, and it is scalar multiple homomorphism since for any $q \in k$, $q\phi(1 \otimes x_i) = qe_i = \phi(q \otimes x_i)$. Thus ϕ is not only A-module map but also k-vector space homomorphism, i.e., linear transformation. And, ϕ is surjective, so does isomorphism since it is finite dimensional vector space map. Thus by exactness, $k \otimes N$ should be zero A-module. By Exercise 2.12, we know that N is finitely generated. Also, $0 = k \otimes N = A/\mathfrak{m} \bigotimes_A N \cong N/\mathfrak{m}N$ implies $N = \mathfrak{m}N$. Since we assume A is local ring, thus $\mathfrak{m}$ is the Jacobson radical, therefore by Nakayama's lemma, $N = 0$.

From this, we know that ϕ is also injective, thus it is isomorphism as a module map. Now we can see that $\{x_i\}$ is linearly independent; suppose that $\sum_{j=1}^n a_j x_j = 0$. Then, $\phi(\sum_{j=1}^n a_j x_j) = 0$ implies $\sum_{j=1}^n a_j e_j = 0$. Since $\{e_i\}$ is canonical basis, $a_j = 0$ for all j. This implies $\{x_j\}$ is linearly independent. Suppose that if $\{x_i\}_{i=1}^m$ with $m < n$ is a generating set of A^n, than add any elements $x_{m+1}, \cdots, x_n$ on the generating set. Then, by above argument, there is a bijection sending $\{x_i\}_{i=1}^n$ to $\{e_i\}_{i=1}^n$ which induces an automorphism on A^n. However, since x_{m+1} is generated by $\{x_i\}_{i=1}^m$, $x_{m+1} = \sum_{j=1}^m a_j x_j$ with nonzero a_is, thus $\phi(x_{m+1}) = \phi(\sum_{j=1}^m a_j x_j)$ implies $\{e_i\}_{i=1}^n$ is not linearly independent, contradiction.

16. Suppose $f : A \to B$ be a ring homomorphism inducing B as A-algebra.

i) $\to$ *ii)*: Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then, by *i)*, $\mathfrak{p}^{ec} = \mathfrak{p}$. By Proposition 3.16, $\mathfrak{p}$ is a contraction of a prime ideal of B, which implies it is image of f^*. Since $\mathfrak{p}$ was arbitrarily chosen, f^* is surjective.

ii) $\to$ *iii)*: Since f^* is surjective, $\mathfrak{m} = f^*(\mathfrak{n})$ for some ideal $\mathfrak{n} \in \mathrm{Spec}(B)$. This implies that $\mathfrak{m} = \mathfrak{n}^c$. Thus, $\mathfrak{m}^e = f(\mathfrak{m}) \subseteq \mathfrak{n} \subsetneq (1)$. Hence $\mathfrak{m}^e \neq (1)$.

iii) $\to$ *iv)*: Let $x \in M$ such that $x \neq 0$. Then $M' := Ax$ is a submodule of M. Since B is flat A-module, notes that $0 \to M' \to M \to M/M' \to 0$ implies $0 \to M'_B \to M_B \to M_B/M'_B \to 0$ is exact. Thus, $M'_B \neq 0$ implies $M_B \neq 0$, thus it suffices to show that $M'_B \neq 0$. Notes that $A \to M' = Ax$ by $a \mapsto ax$ is surjective map, thus $M' \cong A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. (Since M' is nonzero module, $\mathfrak{a} \neq (1)$.) This implies that $B \bigotimes_A A/\mathfrak{a} \cong B/\mathfrak{a}^e$ by Exercise 2.2 stating that $M'_B \cong B \bigotimes_A A/\mathfrak{a} \cong B/\mathfrak{a}B = B/f(\mathfrak{a})B = B/\mathfrak{a}^e$. Since $\mathfrak{a}$ is proper ideal, it is contained in a maximal ideal $\mathfrak{m}$, and by condition iii), $\mathfrak{m}^e \neq (1)$. This implies that $M'_B \neq 0$. Thus, M_B is nonzero.

iv) $\to$ *v)*: Let $M' = \ker(M \to M_B)$ where $M \to M_B$ by $x \mapsto 1 \otimes x$. Then, we have an exact sequence

$$0 \to M' \to M \to M_B$$

and since B is flat,

$$0 \to M'_B \to M_B \to (M_B)_B$$

is also exact. By exercise 2.13 with $N = M_B$ shows that this given map $M_B \to (M_B)_B$ is injective. Thus, $M'_B \to M_B$ is injective zero map, which implies $M'_B = 0$. Now by below claim, $M' = 0$. Thus the map is injective.

Claim XVII. *Let $f : A \to B$ a ring map and B is flat A-algebra, and M is a module. Then, $M_B = B \bigotimes_A M = 0$ implies $M = 0$*

Proof. Suppose M is nonzero. Then, $\exists x \in M$ which is nonzero elements, thus we have

$$0 \to Ax \to M \to M/Ax \to 0.$$

Now by tensoring with B, we get an exact sequence

$$0 \to Bx \to M_B \to B \otimes M/Ax \to 0$$

and from $0 \to Bx \to M_B \to M_B/Bx \to 0$, we know that $B \otimes M/Ax \cong M_B/Bx$. Since M_B is zero, and $Bx \to M_B$ is injective map, this implies $Bx = 0$. Hence $x = 1.x = 0$ implies x is zero element in M as an abelian group, contradiction. ☐

$v) \to i)$: Let $M = A/\mathfrak{a}$ for fixed ideal $\mathfrak{a}$. By Exercise 1.17 i), we have $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$. So suppose that $\mathfrak{a} \subsetneq \mathfrak{a}^{ec}$. Then, $\exists f \in \mathfrak{a}^{ec} \backslash \mathfrak{a}$. Then, $\mathfrak{a}^{ec}/\mathfrak{a}$ is nonzero as a submodule of M. By condition v), $\psi : M \to M_B \cong B/\mathfrak{a}^e$ by $\overline{x} \to 1 \otimes \overline{x} \to \overline{\phi(x)}$ is injective where ϕ is a ring homomorphism $A \to B$ making B as a flat A-algebra. Thus, $\psi(f) \neq 0$, however, $\overline{\phi(f)} = 0$ since $\mathfrak{a}^{ece} = \mathfrak{a}^e$, contradiction.

17. f is flat means that f induces B be a flat A-algebra.

    Let $\phi : N \to M$ be an injective A-module homomorphism. Think about the diagram

$$\begin{tikzcd} N_B \arrow[r, "\phi_B"] \arrow[d] & M_B \arrow[d] \\ N_C \arrow[r, "\phi_C"] & M_C \end{tikzcd}$$

    First of all, ϕ_C is injective since $g \circ f$ is flat. Also notes that

$$C \bigotimes_B (N_B) = C \bigotimes_B (B \bigotimes_A N) \underbrace{\cong}_{\text{Exercise 2.15}} (C \bigotimes_B B) \bigotimes_A N \cong N_C.$$

    Thus, $N_B \to C \bigotimes_B N_B$ by $x \mapsto 1 \otimes x$ is a map defined in Exercise 3.16 v), and since g is faithfully flat, so it is injective. Hence in the above diagram, the bottom, left and right maps are injective. Since they commutes, ϕ_B must be injective.

18. Let $S = A \setminus \mathfrak{p}$, and $T = B \setminus \mathfrak{q}$. Then, $B_\mathfrak{p} = S^{-1}B$, $B_\mathfrak{q} =$ and $f(s) \in T$ induces a map $A_\mathfrak{p} \to B_\mathfrak{q}$ by $a/s \mapsto f(a)/f(s)$. Also, $B_\mathfrak{p} = f(S)^{-1}B$ and from $f(S) \subseteq T$, Exercise 3.3 gives an isomorphism

$$B_\mathfrak{q} = T^{-1}B \cong U^{-1}(f(S)^{-1}B) = U^{-1}(B_\mathfrak{p})$$

    where $U = \{t/1 \in B_\mathfrak{p} : t \in T\}$. Then, we can take a map

$$g : A_\mathfrak{p} \to B_\mathfrak{p} \to U^{-1}(B_\mathfrak{p}) = B_\mathfrak{q} \text{ by } x/s \mapsto f(x)/f(s) \mapsto f(x)/f(s).$$

    Thus this map factor through the map $A_\mathfrak{p} \to B_\mathfrak{q}$. Now notes that $A_\mathfrak{p} \to B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ by Proposition 3.10. In other words, $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-algebra. Also, $B_\mathfrak{q}$ ia a flat $B_\mathfrak{p}$-module by Corollary 3.6. Now notes that $A_\mathfrak{p}$ is a local ring with the maximal ideal $\mathfrak{p}A_\mathfrak{p}$. Then, $g(\mathfrak{p}A_\mathfrak{p}) \subseteq \mathfrak{q}B_\mathfrak{q}$. Thus, $(\mathfrak{p}A_\mathfrak{p})^e \neq (1)$. Hence g is faithfully flat by Exercise 3.16 iii), thus $\mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective, by Exercise 3.16 i).

19. (a) If $\mathrm{Supp}(M) = \emptyset$, then $M_\mathfrak{p} = 0$ for all prime ideal $\mathfrak{p}$, thus $M = 0$ by Proposition 3.8. Conversely, if $M = 0$, then $\mathrm{Supp}(M) = 0$. Now take contrapositive.

(b) Let $\mathfrak{p}$ be a prime ideal, $S = A \setminus \mathfrak{p}$. Then, by 3.4 iii), $(A/\mathfrak{a})_\mathfrak{p} = S^{-1}A/S^{-1}\mathfrak{a}$. First of all, notes that $\mathfrak{p}$ is support of $A/\mathfrak{a}$ if and only if $S^{-1}\mathfrak{a} \neq s^{-1}A$; to see this, since $\mathfrak{a}$ is proper ideal (thus it has no unit of A) and contained in S^c, $a/1 \in S^{-1}A$ are nonunits, thus ideal generated by $S^{-1}\mathfrak{a}$ is proper ideal. It is equivalent to say that $1/1 \neq a/s \in S^{-1}\mathfrak{a}$ for any $a/s \in S^{-1}\mathfrak{a}$. This is equivalent to say that there is no $t, s \in S$ such that $ts = ta \in \mathfrak{a} \cap S$ for any $a \in \mathfrak{a}$. Thus this is equivalent to saying that $\mathfrak{a} \cap S = \emptyset$. This is equivalent to saying that $\mathfrak{a} \subseteq \mathfrak{p}$. This is equivalent to saying that $\mathfrak{p} \in V(\mathfrak{a})$.

(c) For any prime ideal $\mathfrak{p}$, by Proposition 3.3, $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$ is an exact sequence. Thus, $\mathfrak{p} \in \mathrm{Supp}(M)$ if and only if $M_\mathfrak{p}$ is nonzero if and only if either $M'_\mathfrak{p}$ or $M''_\mathfrak{p}$ is nonzero. (If both are zero, the by exactness, $M_\mathfrak{p} = 0$, contradiction.) This is equivalent to say that $\mathfrak{p} \in \mathrm{Supp}(M'_\mathfrak{p}) \cup \mathrm{Supp}(M''_\mathfrak{p})$.

(d) If $\mathfrak{p} \in \bigcup \mathrm{Supp}(M_i)$, then the inclusion $M_i \to M$ induces another injection $(M_i)_\mathfrak{p} \to M_\mathfrak{p}$ by Proposition 3.3. Hence, $M_\mathfrak{p}$ is nonzero, thus $\mathfrak{p} \in \mathrm{Supp}(M)$. In other direction, we cannot use corollary 3.4 directly, since the sum maybe infinite.

To see this, notes that $\bigoplus M_i \to \sum M_i = M$ has natural injective map. We claim that

Claim XVIII. *Tensor product commutes with arbitrary direct sums. This induces localization commutes with arbitrary direct sum, i.e.,*

$$S^{-1}(\bigoplus M_i) \cong \bigoplus S^{-1}M_i$$

for arbitrary direct sum (coproduct.)

By this claim the induced map by Proposition 3.3 $\bigoplus (M_i)_\mathfrak{p} \cong (\bigoplus M_i)_\mathfrak{p} \to (\sum M_i)_\mathfrak{p} = M_\mathfrak{p}$ is also injective. Hence, if $(M_i)_\mathfrak{p} = 0$ for all i, then $M_\mathfrak{p} = 0$. Its contrapositive says that if $\mathfrak{p} \in \mathrm{Supp}(M)$, then $\mathfrak{p} \in \bigcup \mathrm{Supp}(M_i)$.

Proof. Proof of the claim Let $\bigoplus_{i \in I} M_i$ be direct sum of an arbitary module. Now define a map

$$f : N \times \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} N \otimes M_i \text{ by } (n, (m_i)_{i\in I}) \mapsto (n \otimes m_i)_{i \in I}.$$

First of all, it is bilinear since given tensor operation is bilinear. Hence, by the universal property, this map induces a map $\tilde{f} : N \bigotimes_A \bigoplus_{i\in I} M_i \to \bigoplus_{i \in I} N \otimes M_i$ by $n \otimes (m_i)_{i \in I} \mapsto (n \otimes m_i)_{i\in I}$. Then we claim that it is isomorphism. To see this, $\tilde{f}((n \otimes m_i)_{i\in I}) = 0$ implies $n \otimes m_i = 0$ for all i, thus $n = 0$ or all m_is are zeo, which implies $n \otimes (m_i)_{i\in I} = 0$. Also, it is surjective in clear sense.
Now we can apply this fact to show that S^{-1} commutes with arbitrary direct sum.

$$S^{-1}(\bigoplus M_i) \underbrace{\cong}_{\text{Prop 3.5}} S^{-1}A \bigotimes_A \bigoplus M_i) \cong \bigoplus S^{-1}A \bigotimes_A M_i \underbrace{\cong}_{\text{Prop 3.5}} \bigoplus S^{-1}M_i.$$

$\square$

We just leave a claim which will be helpful for future. (Maybe.)

Claim XIX. *For any multiplicative subset S of A, we can give a direct system structure on S, so that $S^{-1}M$ is isormophic to $\varinjlim_{s\in S} M$.*

Proof. First of all, define order on S. Let $s \le t$ if $\exists u \in S$ such that $t = us$. Then S is directed set, since for any $s, t \in S$, $s \le st$ and $t \le st$. Also, for given A-module M, for each $s \in S$ let M_s be a localization of M by $\{1, s, s^2, \cdots\}$. Then for each pair $i, j \in S$ with $i \le j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism defined by $a/i^k \mapsto at^k/j^k$ where $t \in S$ is an element satisfying $j = it$. First of all, it is well-defined, since for any two t_1, t_2 satisfying $j = it_1 = it_2$, $i^k(j^k at_1^k - j^k at_2^k) = j^k a((it_1)^k - (it_2)^k) = j^k a(j^k - j^k)$ implies $at_1^k/j^k = at_2^k/j^k$. Then, it satisfies that

i. μ_{ii} is identity mapping by letting $t = 1$.

ii. $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$. To see this, let $t, q \in S$ such that $j = it$, $k = qj$. Then,

$$\mu_{ik}(a/i^n) = a(q^n t^n)/k^n = \mu_{jk}(at^n/j^n) = \mu_{jk} \circ \mu_{ij}(a/i^n).$$

Thus, M_i with homomorphisms μ_{ij} form a direct system $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set S, by following Exercise 2.14. Hence, it admits a direct limit $\varinjlim_{s\in S} M_s$ by construction of Exercise 2.14. Let $\mu_s : M_s \to \varinjlim_{s\in S} M_s$ be restriction of projection map $\bigoplus_{s\in S} M_s \to \varinjlim_{s\in S} M_s$. Notes that if $0 \in S$, then 0 is the maximal element, since for any $s \in S$, $0s = 0$ implies $s \leq 0$. Thus, the direct limit is zero, since for any element $\mu_s(a/s^k)$ (by Exercise 2.15), $\mu_s(x_s) = \mu_0(a0^k/s^k) = \mu_0(0/1) = 0$ since ring homomorphism sends 0 to 0.

Now take a map $\alpha_s : M_s \to S^{-1}M$ by $a/s^k \mapsto a/s^k$. It is well-defined homomorphism since if $a/s^k = a'/s^q$ in M_s, then $\exists s^t$ such that $s^t a' s^q - s^t a s^q = 0$ thus $a' s^{t+q} = a s^{t+q}$. Thus, $a/s^k = a'/s^q$ in $S^{-1}M$ since $s^t \in S$. Also, for $i \leq j$ there exists $u \in S$ such that $ui = j$, $\alpha_j \circ \mu_{ij}(a/i^k) = \alpha_j(au^k/j^k) = au^k/j^k$ and $au^k/j^k = \alpha_i(a/i^k)$ in $S^{-1}M$ since $1(au^k i^k - aj^k) = 0$. Hence it satisfies $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. By Exercise 2.16, there exists a unique homomorphism $\alpha : \varinjlim_{s\in S} M_s \to S^{-1}M$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Now we want to show that α is isomorphism. By Exercise 2.15, every element of $\varinjlim M_s$ can be denoted by $\mu_i(a/i^k)$. Then, to see α is injective, suppose $\alpha \circ \mu_i(a/i^k) = 0$. This implies $\alpha_i(a/i^k) = 0$, hence $a/i^k = 0$ in $S^{-1}M$. Thus $\exists t \in S$ such that $ta = 0$. Thus, let $q = ti$. Then $q \geq i$, thus $\mu_i(a/i^k) = \mu_q \circ \mu_{iq}(a/i^k) = \mu_q(at^k/q^k) = \mu_q(0/q^k) = 0$. Also, this map is surjective, since for any $a/s \in S^{-1}M$, $\alpha \circ \mu_s(a/s) = \alpha_s(a/s) = a/s$. Done.

$\square$

(e) Let $\{x_i\}_{i=1}^n$ be a set generating M. Then, $Ax_i \cong A/\mathfrak{a}_i$ for some proper ideal $\mathfrak{a}_i$ (by taking a canonical map $A \to Ax_i$). Thus, from $M = \sum_{i=1}^n Ax_i$,

$$\mathrm{Supp}(M) \underbrace{=}_{\text{iv)}} \bigcup \mathrm{Supp}(Ax_i) \underbrace{=}_{\text{Prop 3.9}} \bigcup \mathrm{Supp}(A/\mathfrak{a}_i) \underbrace{=}_{\text{ii)}} \bigcup_{i=1}^n V(\mathfrak{a}_i) \underbrace{=}_{\text{Exercise 1.15 iv)}} V(\bigcap_{i=1}^n \mathfrak{a}_i).$$

Now if $a \in A$ annihilates M, then the map $A \to Ax_i$ sends a to 0, hence $a \in \bigcap_{i=1}^n \mathfrak{a}_i$. Conversely, any elements in $\bigcap_{i=1}^n \mathfrak{a}_i$ goes zero when we multiply it with any generators. Thus this set is in $\mathrm{Ann}(M)$. This implies that $\bigcap_{i=1}^n \mathfrak{a}_i = \mathrm{Ann}(M)$.

(f) Suppose that $\mathfrak{p}$ is not in $\mathrm{Supp}(M \bigotimes_A N)$. Then, $(M \otimes N)_\mathfrak{p} = 0$, which implies $M_\mathfrak{p} \otimes N_\mathfrak{p} = 0$ by Proposition 3.7. By Exercise 2.3, $M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0$, thus $\mathfrak{p} \notin \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. Conversely, if $\mathfrak{p} \notin \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$, then either $M_\mathfrak{p}$ or $N_\mathfrak{p}$ is zero. By Exercise 2.3, $M_\mathfrak{p} \otimes N_\mathfrak{p} = 0$. Hence by Propositon 3.7, $(M \otimes N)_\mathfrak{p} = 0$, which implies $\mathfrak{p} \notin \mathrm{Supp}(M \otimes N)$. Now take contrapositive what we get.

(g) $M/\mathfrak{a}M \cong A/\mathfrak{a} \bigotimes_A M$ by Execise 2.2, so

$$\mathrm{Supp}(M/\mathfrak{a}M) \underbrace{=}_{\text{vi)}} \mathrm{Supp}(A/\mathfrak{a}) \cap \mathrm{Supp}(M) \underbrace{=}_{\text{by v)}} V(\mathfrak{a}) \cap V(\mathrm{Ann}(M)) \underbrace{=}_{\text{by Exercise 1.15 iii)}} V(\mathfrak{a} \cup \mathrm{Ann}(M)).$$

Notes that every ideal containing $\mathfrak{a} \cup \mathrm{Ann}(M)$ contains $\mathfrak{a} + \mathrm{Ann}(M)$ since it is an ideal generated by $\mathfrak{a} \cup \mathrm{Ann}(M)$. Hence, we get the desired result.

(h) As a B-module, $(B \bigotimes_A M)_\mathfrak{q} \cong B_\mathfrak{q} \bigotimes_B (B \bigotimes_A M)$ by Proposition 3.5. By Exercise 2.15, with the property that B is a bimodule, we know that $B_\mathfrak{q} \bigotimes_B (B \bigotimes_A M) \cong (B_\mathfrak{q} \bigotimes_B B) \bigotimes_A M$ and by Proposition 3.5, $(B_\mathfrak{q} \bigotimes_B B) \cong B_\mathfrak{q}$, hence

$$(B \bigotimes_A M)_\mathfrak{q} \cong B_\mathfrak{q} \bigotimes_A M.$$

Also, notes that $B_\mathfrak{q}$ has a $A_\mathfrak{p}$-module structure such that for $a/s \in A_\mathfrak{p}$, define $(a/s).(b/t) := f(a)/f(s) \cdot b/t$. This is well-defined action since if $a/s = a'/s'$, then $\exists t \in S$ such that $ts'a = tsa'$, hence $f(a)/f(s) = f(a')/f(s')$ since $f(t) \in A - \mathfrak{q}$ because $\mathfrak{p}$ is preimage of $\mathfrak{q}$. Hence, we have

$$(B \bigotimes_A M)_\mathfrak{q} \cong B_\mathfrak{q} \bigotimes_A M \cong (B_\mathfrak{q} \bigotimes_{A_\mathfrak{p}} A_\mathfrak{p}) \bigotimes_A M$$

And as we already know, $A_\mathfrak{p}$ is a bimodule over $A_\mathfrak{p}$ and A, thus

$$(B \bigotimes_A M)_\mathfrak{q} \cong (B_\mathfrak{q} \bigotimes_{A_\mathfrak{p}} A_\mathfrak{p}) \bigotimes_A M \cong B_\mathfrak{q} \bigotimes_{A_\mathfrak{p}} (A_\mathfrak{p} \bigotimes_A M) \cong B_\mathfrak{q} \bigotimes_{A_\mathfrak{p}} M_\mathfrak{p}$$

by Exercise 2.15 and Proposition 3.5.
Now suppose that $\mathfrak{p} \notin \mathrm{Supp}(M)$. Then, $M_\mathfrak{p} = 0$, this implies $(B \bigotimes_A M)_\mathfrak{q} \cong B_\mathfrak{q} \bigotimes_{A_\mathfrak{p}} M_\mathfrak{p} = 0$ by Exercise 2.3. Hence, $\mathfrak{q} \notin \mathrm{Supp}(M_B)$.
Conversely, suppose $\mathfrak{q} \notin \mathrm{Supp}(M_B)$. Then $0 = (B \bigotimes_A M)_\mathfrak{q} \cong B_\mathfrak{q} \bigotimes_{A_\mathfrak{p}} M_\mathfrak{p}$. Let $x_1, \cdots, x_n$ be generators of M over A. Then $(1/1) \otimes (x_i/1) = 0$. By Corollary 2.13 there exists a finitely generated $A_\mathfrak{p}$-submodule N_i of $B_\mathfrak{q}$ and M_i of $M_\mathfrak{p}$such that $N_i \bigotimes_{A_\mathfrak{p}} M_i$ is a finitely generated submodule such that $(1/1) \otimes (x_i/1) = 0$. Now notes that $N_i \bigotimes_{A_\mathfrak{p}} M_\mathfrak{p}$ is also a finite $A_\mathfrak{p}$-submodule of $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} M_\mathfrak{p}$ containing $\sum_{i=1}^n N_i \bigotimes_{A_\mathfrak{p}} M_i$, since it is generated by generators of N_i and its tensor product with $\{x_i\}$s, which are finite.
Now $\sum_{i=1}^n (N_i \bigotimes_{A_\mathfrak{p}} M_q)$ is a finite $A_\mathfrak{p}$-submodule such that $(1/1) \otimes (x_i/1) = 0$ for all x_i. Since the sum is finite, so it is actually a direct sum, hence using Proposition 2.14 we can rewrite it as $N \bigotimes_{A_\mathfrak{p}} M_\mathfrak{p}$ where $N = \sum_{i=1}^n N_i$, which is still finitely generated. Then, $N \bigotimes_M M_\mathfrak{p} = 0$ since $(1/1) \otimes (x/1) = 0$ for all $x \in M$, since each x is generated by $(1/1) \otimes x_i/1$. Thus $N = 0$ or $M_\mathfrak{p} = 0$ by Exercise 2.3. Since N has $1/1$, it is nonzero, this implies $M_\mathfrak{p} = 0$.

20. (a) Let $\mathfrak{p} = \mathfrak{q}^c$ for some $\mathfrak{q}$. Then, $f^*(\mathfrak{q}) = f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Hence if LHS true then f^* is surjective. Conversely, if f^* is surjective, for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathfrak{p} = f^*(\mathfrak{q}) = \mathfrak{q}^c$, done.

    (b) Suppose the LHS. Let $\mathfrak{q} = \mathfrak{p}^e$. Then, $f^*(\mathfrak{q}) = \mathfrak{p}^{ec} \supseteq \mathfrak{p}$. Thus, suppose $f^*(\mathfrak{q}) = f^*(\mathfrak{q}')$. Then, $\mathfrak{p}^{ec} = (\mathfrak{p}')^{ec}$, where $\mathfrak{q}' = (\mathfrak{p}')^e$, this implies $\mathfrak{p}^{ece} = (\mathfrak{p}')^{ece}$, which implies $\mathfrak{p}^e = (\mathfrak{p}')^e$ by Proposition 1.17, which implies $\mathfrak{q} = \mathfrak{q}'$.

    Converse is not true. Let $B = A[x]/(x^2)$ for any ring A. Then, $\pi : B \to A$ by $f \mapsto f(0)$ has kernel (x). Thus, for any prime ideal $\mathfrak{p}$ of A, its inverse is a prime ideal of B containing x. And for any prime ideal in B, it contains $0 = x^2$, thus x is in the prime ideal. Thus all prime ideals in B are contraction of A. Now let $\phi : A \to B$ canonical injection. Then $\mathfrak{p}^e$ along ϕ is not a prime ideal, since $\mathfrak{p}^e = \langle \mathfrak{p} \rangle = \mathfrak{p}A[x]/(x^2) = \{a + bx \in B : a, b \in \mathfrak{p}\}$, thus $x \notin \mathfrak{p}^e$. (If it is in, then $\mathfrak{p}$ contains 1, contradiction.) Also, $\pi^{-1}(\mathfrak{p})$ contains $\mathfrak{p}^e$. Hence no extended ideals are prime, and no prime ideal $\pi^{-1}(\mathfrak{p})$ is extended. However, ϕ induces a bijection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ since $\phi^{-1}(\pi^{-1}(\mathfrak{p})) = (\pi \circ \phi)^{-1}(\mathfrak{p}) = 1_A^{-1}(\mathfrak{p}) = \mathfrak{p}$.

21. (a) By Proposition 3.11, we have a one-to-one correspondence between $\mathrm{Spec}(S^{-1}A)$ and $S^{-1}X := \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$ given by ϕ^* (contraction) and extension of prime ideal. In particular, by 3.11 ii), if $\mathfrak{p} \in S^{-1}X$,

    $$\mathfrak{p}^{ec} = \bigcup_{s \in S} (\mathfrak{p} : s) = \mathfrak{p}$$

    where the last equality comes from the observation that $s \notin \mathfrak{p}$, thus $as \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ for all $a \in A$. This implies $(\mathfrak{p} : s) = \mathfrak{p}$ for all $s \in S$. (Or just use the one-to-one correspondence to see that $\exists \mathfrak{q} \in Y$ such that $\mathfrak{q}^c = \mathfrak{p}$, hence $\mathfrak{p}^{ec} = \mathfrak{q}^{cec} = \mathfrak{q}^c = \mathfrak{p}$.) By the same argument, if $\mathfrak{q} \in Y$, then $\mathfrak{q} = \mathfrak{p}^e$ by 1-1 correspondence, thus

$$\mathfrak{q}^{ce} = \mathfrak{p}^{ece} = \mathfrak{p}^e = \mathfrak{q}$$

Now let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(S^{-1}A)$, then for any $f \in S$, we want to show that $\phi^{*-1}(X_f) = Y_{\phi(f)}$. Suppose that $\mathfrak{p} \in Y_{\phi(f)}$. Then $\phi(f) \notin \mathfrak{p}$, thus $f \notin \mathfrak{p}^c = \phi^*(\mathfrak{p})$. Hence $\phi^*(\mathfrak{p}) \in X_f$. Thus $\mathfrak{p} \in \phi^{*-1}(X_f)$. Thus by Exercise 1.21 i), ϕ^* is continuous.

Also, to show homeomorphism, we want to show that ϕ^* is also closed map. Let $V(\mathfrak{a})$ be closed set in Y for some ideal $\mathfrak{a}$. Then,

$$\phi^*(V(\mathfrak{a})) = \{\mathfrak{p}^c : \mathfrak{p} \in V(\mathfrak{a})\} = \{\mathfrak{q} \in S^{-1}X : \mathfrak{q} \supseteq \phi^{-1}(\mathfrak{a})\} = V(\phi^{-1}(\mathfrak{a})) \cap S^{-1}X.$$

Thus, if $\mathfrak{a}$ is proper, then $\phi^{-1}(\mathfrak{a})$ is also a proper ideal, thus $V(\phi^{-1}(\mathfrak{a})) \cap S^{-1}X$ is closed in $S^{-1}X$ as subspace topology. This shows that ϕ is closed map, thus homeomorphism.

Now for the last statement, let $S = \{1, f, f^2, \cdots\}$. Then, if $\mathfrak{p} \in X_f$, then $f \notin \mathfrak{p}$, thus $S \cap \mathfrak{p} = \emptyset$ since $\mathfrak{p}$ is prime, hence $\mathfrak{p} \in S^{-1}X$ by one-to-one correspondence from Proposition 3.11. Conversely, if $\mathfrak{p} \in S^{-1}X$, then $\mathfrak{p} \cap S = \emptyset$, this implies $f \notin \mathfrak{p}$, hence $\mathfrak{p} \in X_f$. This shows that $\mathrm{Im}(\mathrm{Spec}(A_f) = S^{-1}X = X_f$.

(b) What we want to show is that below diagram commutes,

$$\begin{tikzcd} S^{-1}Y \arrow[r, "g^*"] \arrow[d, "\pi_B^{*-1}(\cong)"'] & S^{-1}X \arrow[d, "\pi_A^{*-1}(\cong)"] \\ \mathrm{Spec}(S^{-1}B) \arrow[r, "S^{-1}f^*"] & \mathrm{Spec}(S^{-1}A) \end{tikzcd}$$

where $S^{-1}Y$ is image of $\mathrm{Spec}(S^{-1}B)$ in $\mathrm{Spec}(B)$ and $S^{-1}X$ is image of $\mathrm{Spec}(S^{-1}A)$ in $\mathrm{Spec}(A)$, and $g^* = f^*|_{S^{-1}Y}$., and $S^{-1}f : S^{-1}A \to S^{-1}B$ by $a/s \mapsto f(a)/f(s)$. Homeomorphism is clear by above exercise. So it suffices to show that it commutes.

Let $\mathfrak{p} \in S^{-1}Y$. Then, $g^*(\mathfrak{p}) = f^{-1}(\mathfrak{p})$. Also, $S \cap f^{-1}(\mathfrak{p}) = \emptyset$ otherwise $f(s) \in \mathfrak{p}$, contradicting definition of $S^{-1}Y$. Hence, $\pi_A^{*-1} \circ g^*(\mathfrak{p})$ is a prime ideal $f^{-1}(\mathfrak{p})^e$, which is an ideal generated by $\{a/1 : a \in f^{-1}(\mathfrak{p})\}$. Conversely, localization on B by $f(S)$ maps $\mathfrak{p}$ into $\{b/1 : b \in \mathfrak{p}\}$, so an ideal generated by extension of this set is $\pi_A^{*-1}(\mathfrak{p})$. Now $S^{-1}f^* \circ \pi_A^{*-1}(\mathfrak{p}) := \{a/s : f(a)/f(s) \in \pi_A^{*-1}(\mathfrak{p})\}$. Now we claim that those are equal.

Let $a/s \in S^{-1}f^* \circ \pi_A^{*-1}(\mathfrak{p})$, Then, $f(a)/f(s) \in \pi_A^{*-1}(\mathfrak{p})$. Thus, $f(a) \in \mathfrak{p}$, otherwise, since $f(s) \notin \mathfrak{p}$ for any $s \in S$, $f(a)/f(s)$ has no representation such that numerator is in $\mathfrak{p}$. Hence, $a \in f^{-1}(\mathfrak{p})$, thus $a/s = a/1 \cdot 1/s$ is in $f^{-1}(\mathfrak{p})^e$. Conversely, if $a/s \in f^{-1}(\mathfrak{p})^e$, then $a/s = \sum b_i/c_i \cdot a_i/1$ for some $b_i/c_i \in S^{-1}A$ and $a_i \in f^{-1}(\mathfrak{p})$, $a \in f^{-1}(\mathfrak{p})$ since every term of numerator in forms of the reduction of common denominators are in $f^{-1}(\mathfrak{p})$. Hence, $f(a) \in \mathfrak{p}$, therefore $f(a)/f(s) = f(a)/1 \cdot 1/f(s) \in \pi_A^{*-1}(\mathfrak{p})$, an extension of $\mathfrak{p}$. This shows that $a/s \in S^{-1}f^* \circ \pi_A^{*-1}(\mathfrak{p})$. Hence

$$S^{-1}f^* \circ \pi_A^{*-1}(\mathfrak{p}) = f^{-1}(\mathfrak{p})^e.$$

Hence the above diagram commutes.

(c) Let $\mathfrak{q} \in V(\mathfrak{b})$. Then, $\mathfrak{a} \subseteq \mathfrak{a}^{ec} = \mathfrak{b}^c \subseteq \mathfrak{q}^c = f^*(\mathfrak{q})$. This implies $f^*(V(\mathfrak{b})) \subseteq V(\mathfrak{a})$. If we let $\pi_A : A \to A/\mathfrak{a}, \pi_B : B \to B/\mathfrak{b}$, then $\tilde{f} \circ \pi_A(a) = \tilde{f}(\overline{a} + \mathfrak{a}) = \overline{f(a)} + \mathfrak{b} = \pi_B \circ f(a)$ for any $a \in A$. Hence, $\tilde{f} \circ \pi_A = \pi_B \circ f$. This induces $(\pi_A)^* \circ \tilde{f}^* = f^* \circ \pi_B^*$, which shows the below map commutes.

$$\begin{tikzcd} \mathrm{Spec}(B/\mathfrak{b}) \arrow[r, "\tilde{f}^*"] \arrow[d, "\pi_B^*"'] & \mathrm{Spec}(A/\mathfrak{a}) \arrow[d, "\pi_A^*"] \\ V(\mathfrak{b}) \arrow[r, "f^*"] & V(\mathfrak{a}) \end{tikzcd}$$

Since each π_A^*, π_B^* are homeomorphism, so we can say that $\tilde{f}^*$ is restriction of f^* on $V(\mathfrak{b})$.

(d) If we use result of ii) and iii), then we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \arrow[d] & \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \arrow[l, "\overline{f_p}^*"'] \arrow[d] \\
\operatorname{Spec}(A_{\mathfrak{p}}) \arrow[d] & \operatorname{Spec}(B_{\mathfrak{p}}) \arrow[l, "f_p^*"'] \arrow[d] \\
\operatorname{Spec}(A) & \operatorname{Spec}(B) \arrow[l, "f^*"']
\end{tikzcd}$$

By iii), $\operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is injective map, and by ii) $\operatorname{Spec}(A_{\mathfrak{p}}) \to \operatorname{Spec}(A)$ is injective map. (Similarly for the right two maps.) Notes that $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ by definition. Thus, $\operatorname{Spec}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ is singleton, hence its image on $\operatorname{Spec}(A_{\mathfrak{p}})$ is $V(\mathfrak{p}A_{\mathfrak{p}})$ by iii), which is $\{\mathfrak{p}A_{\mathfrak{p}}\}$ since it is the maximal ideal in $A_{\mathfrak{p}}$. Also, image of $V(\mathfrak{p}A_{\mathfrak{p}})$ in $\operatorname{Spec}(A)$ is $\{\mathfrak{q} \in S^{-1}\operatorname{Spec}(A) : \mathfrak{q} \supseteq \mathfrak{p}\} = \{\mathfrak{p}\}$, since if $\mathfrak{q} \in S^{-1}\operatorname{Spec}(A)$, then by definition $\mathfrak{q} \cap S = \emptyset$, thus $\mathfrak{q} \subseteq A - S = \mathfrak{q}$. Thus, if we send $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ along $\overline{f}_p^*$, then it is goes zero map, and its image on $\operatorname{Spec}(A)$ is $\{\mathfrak{p}\}$, which is equal to sending it using $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \to \operatorname{Spec}(B_{\mathfrak{p}}) \to \operatorname{Spec}(B) \to \operatorname{Spec}(A)$. Thus, $f^{*-1}(\mathfrak{p})$ contains image of $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. Also, if $\mathfrak{q} \in f^{*-1}(\mathfrak{p})$, then $f^*(\mathfrak{q}) = \mathfrak{p}$, thus $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ implies $f(\mathfrak{p}) \subseteq \mathfrak{q}$, hence $\mathfrak{q} \cap f(S) = \emptyset$. Hence, $\mathfrak{q}$ is in the image of $\operatorname{Spec}(B_{\mathfrak{p}})$ where $B_{\mathfrak{p}} = f(S)^{-1}B$. Let $(\mathfrak{q})_{\mathfrak{p}}$ be such a preimage of $\mathfrak{q}$. Since the map is injective, it is actually the preimage of $\mathfrak{q}$. And, since $f(\mathfrak{p}) \subseteq \mathfrak{q}$, $\mathfrak{q}_{\mathfrak{p}}$ contains $f(\mathfrak{p})B_{\mathfrak{p}}$, thus $\mathfrak{q}_{\mathfrak{p}} \in V(f(\mathfrak{p})B_{\mathfrak{p}})$, and by iii), $\mathfrak{q}_{\mathfrak{p}}$ is image of an element in $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. This implies that $f^{*-1}(\mathfrak{p}) = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ as a subspace of $\operatorname{Spec}(B)$, thus the above injections homeomorphic to its image induces natural homeomorphism.

Now to see that $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) = \operatorname{Spec}(k(\mathfrak{p}) \bigotimes_A B)$, let T be image of S along the map $A \to A/\mathfrak{p}$. Then, since $S = A \backslash \mathfrak{p}$, $T = (A/\mathfrak{p}) \backslash \{0\}$. Thus, according to Exercise 3.4, $T^{-1}(A/\mathfrak{p}) \cong S^{-1}(A/\mathfrak{p}) = (A/\mathfrak{p})_{\mathfrak{p}}$

$$\begin{aligned}
k(\mathfrak{p}) \bigotimes_A B \cong (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \bigotimes_A B &\underbrace{\cong}_{\text{Prop 3.4 iii)}} (A/\mathfrak{p})_{\mathfrak{p}} \bigotimes_A B \underbrace{\cong}_{\text{Exercise 3.4}} T^{-1}(A/\mathfrak{p}) \bigotimes_A B \\
&\underbrace{\cong}_{\text{Prop 3.5}} (T^{-1}(A/\mathfrak{p}) \bigotimes_{A/\mathfrak{p}} A/\mathfrak{p}) \bigotimes_A B \underbrace{\cong}_{\text{Exercise 2.15}} T^{-1}(A/\mathfrak{p}) \bigotimes_{A/\mathfrak{p}} (A/\mathfrak{p} \bigotimes_A B) \\
&\underbrace{\cong}_{\text{Exercise 2.2}} T^{-1}(A/\mathfrak{p}) \bigotimes_{A/\mathfrak{p}} B/\mathfrak{p}B \underbrace{\cong}_{\text{Prop 3.5}} T^{-1}(B/\mathfrak{p}B) \underbrace{\cong}_{\text{Exercise 3.4}} S^{-1}(B/\mathfrak{p}B) \\
&\underbrace{\cong}_{\text{Prop 3.4 iii)}} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}.
\end{aligned}$$

This shows equality.

22. Notes that $\mathfrak{q} \in \operatorname{Spec}(A_{\mathfrak{p}})$ as a homeomorphic subspace of $\operatorname{Spec}(A)$ if and only if $\mathfrak{q}$ is contained in $\mathfrak{p}$ if and only if $\mathfrak{q} \cap S = \emptyset$ if and only if $\mathfrak{q} \in \bigcap_{f \in S} X_f$. Hence $\operatorname{Spec}(A_{\mathfrak{p}}) = \bigcap_{f \in S} X_f$. Thus it suffices to show that $\bigcap_{f \in S} X_f$ is the intersection of all open neighborhoods of $\mathfrak{p}$.

    Also observe that If U contains $\mathfrak{p}$, then since $\{X_f\}_{f \in A}$ forms a base of Zariski topology by Exercise 1.17, U is union of X_fs. Hence, at least one of X_f in the union contains $\mathfrak{p}$, this implies that $\exists f \in S$ such that X_f is in a union representation of U. Thus, $U \supseteq \bigcap_{f \in S} X_f$, thus intersectionof all neighborhoods of $\mathfrak{p}$ contains $\bigcap_{f \in S} X_f$. Conversely, if $\mathfrak{q} \in \bigcap_{f \in S} X_f$, then it is contained in any open neighborhoood of $\mathfrak{p}$ since U contain at least one of X_f with $f \in S$ as its subset. Thus, $\mathfrak{q}$ is in the intersection of all open neighborhood of $\mathfrak{p}$. Thus, the intersection is $\bigcap_{f \in S} X_f$.

23. (a) To show this, suppose $U = X_f = X_g$. Then, it suffices to show that $A_f \cong A_g$ as a ring. Notes that $X_f = X_g$ gives us $r((f)) = r((g))$ by Exercise 1.17 iv). Hence, $f^n = hg$ and $g^m = h'f$ for some $h, h' \in A$, $n, m \in \mathbb{N}$. Now let $\phi_f : A \to A_f, \phi_g : A \to A_g$. Then, $\phi_f(g) = g/1$ is invertible since $h/f^n \cdot g/1 = 1/1$. Hence, for given $S_g = \{1, g, g^2, \cdots\}$, $\phi_f(s)$ is unit in A_f for all

$s \in S_g$. Thus by Proposition 3.1. (universal property), there exists a unique ring homormophism $\psi_{gf} : A_g \to A_f$ such that $\phi_f = \psi_{gf} \circ \phi_g$. By the same argument, we can get the unique ring homomorphism$\psi_{fg} : A_f \to A_g$ such that $\phi_g = \psi_{fg} \circ \phi_f$. We can draw a commutative diagram

$$\begin{tikzcd}
 & A \arrow[dl, "\phi_f"'] \arrow[dr, "\phi_g"] & \\
A_f \arrow[rr, shift left, "\psi_{fg}"] & & A_g \arrow[ll, shift left, "\psi_{gf}"]
\end{tikzcd}$$

From this diagram, we can get

$$\phi_f = \psi_{fg} \circ \psi_{gf} \circ \phi_f.$$

Then, for any $s \in S_f$, $\psi_{fg} \circ \psi_{gf} \circ \phi_f(s) = \phi_f(s)$ is unit. Hence by Proposition 3.1., there exists a unique homomorphism $\varphi_f : A_f \to A_f$ such that $\psi_{fg} \circ \psi_{gf} \circ \phi_f = \varphi_f \circ \phi_f$. Thus, by uniqueness, $\varphi_f = \psi_{fg} \circ \psi_{gf}$. Also, since $\psi_{fg} \circ \psi_{gf} \circ \phi_f = \phi_f = 1_{A_f} \circ \phi_f$, by uniqueness of φ_f, $\varphi_f = 1_{A_f}$. This induces that $\psi_{fg} \circ \psi_{gf} = 1_{A_f}$. By the same argument on the other side, $\psi_{gf} \circ \psi_{fg} = 1_{A_g}$. Thus $A_g \cong A_f$.

(b) Let $U = X_f$, $U' = X_g$ such that $U' \subseteq U$. Then, $V(f)^c \subseteq V(g)^c$ implies $V(f) \supseteq V(g)$, thus $r((f)) \supseteq r((g))$. Hence, $g \in r((f))$, therefore $\exists n \in \mathbb{N}$ and $u \in A$ such that $g^n = uf$.

Define $\rho : A(U) \to A(U')$ by $a/f^m \mapsto au^n/g^{mn}$. To see ρ is well-defined, let $b/f^k = a/f^m$. Then $\exists f^q$ such that $f^q(af^k - bf^m)$. Hence, for given $\rho(b/f^k) = bu^k/g^{nk}$ and $\rho(a/f^m) = au^m/g^{nm}$,

$$f^q \left(au^m g^{nk} - bu^k g^{nm}\right) = f^q \left(au^m u^k f^k - bu^k u^m f^m\right) = u^{m+k} f^q(af^k - bf^m) = 0$$

Thus ρ is well-defined. Now to see ρ depends only on U and U', Let $U = X_f = X_{f'}, U' = X_g = X_{g'}$. Then, there exists $n, n' \in \mathbb{N}$ and $u, u' \in A$ such that $g^n = uf, (g')^{n'} = u'f$. Hence, we can define $\rho(a/f^m) = au^m/g^{mn}$ and $\rho'(a/f^m) = a(u')^{n'}/g^{mn'}$. So we have a diagram

$$\begin{tikzcd}
A_f \arrow[r, "\rho"] \arrow[d, "\psi_{ff'}"'] & A_g \arrow[d, "\psi_{gg'}"] \\
A_{f'} \arrow[r, "\rho'"] & A_{g'}
\end{tikzcd}$$

where $\psi_{ff'}, \psi_{gg'}$ are isomorphism induced by the universal property in the above construction in i). It suffices to show that this diagram commutes. To see this, first, we show that actually, $\phi_g = \rho \circ \phi_f$. To see this, notes that $\phi_g(f) = f/1$ is unit since $f \cdot u/g^n = 1$. Thus, there exists a unique homomorphism ρ'' such that $\phi_g = \rho \circ \phi_f$, by Proposition 3.1. Now notes that $\phi_g(a) = a/1$, and $\rho \circ \phi_f(a) = \rho(a/1) = a/1$ by definition, thus by uniqueness, $\rho = \rho''$. Likewise, we can show that ρ' is unique homomorphism satisfying $\phi_{g'} = \rho' \circ \phi_{f'}$. Thus, below diagram commutes for each triangle. It suffices to show that outer square commutes.

$$\begin{tikzcd}
A_f \arrow[rr, "\rho"] \arrow[dd, "\psi_{ff'}"'] & & A_g \arrow[dd, "\psi_{gg'}"] \\
 & A \arrow[ul, "\phi_f"] \arrow[ur, "\phi_g"'] \arrow[dl, "\phi_{f'}"'] \arrow[dr, "\phi_{g'}"] & \\
A_{f'} \arrow[rr, "\rho'"] & & A_{g'}
\end{tikzcd}$$

First of all,

$$\psi_{gg'} \circ \rho \circ \phi_f = \psi_{gg'} \circ \phi_g = \phi_{g'} \text{ and } \rho' \circ \psi_{ff'} \circ \phi_f = \rho \circ \phi_{f'} = \phi_{g'}$$

by commuting triangle. And each map sends g' to unit, thus there exists unique map $\varphi : A_{g'} \to A_{g'}$ such that $\psi_{gg'} \circ \rho \circ \phi_f = \varphi \circ \phi_{g'}$ and $\rho' \circ \psi_{ff'} \circ \phi_f = \varphi \circ \phi_{g'}$. Since each righthandside of equations are equal to $\phi_{g'}$, so $\varphi = 1_{A_{g'}}$, and also

$$\psi_{gg'} \circ \rho \circ \phi_f = \rho' \circ \psi_{ff'} \circ \phi_f.$$

Now we claim that ϕ_f is epimorphism. (Notes that epimorphism is not the same as surjection in a category of ring.) Hence it is right cancelable, which implies

$$\psi_{gg'} \circ \rho = \rho' \circ \psi_{ff'}.$$

Claim XX. *Canonical map of localization and quotients are epimorphism.*

Proof. Let $f : A \to A/\mathfrak{a}$ be canonical projection. Then, suppose that $g, h : A/\mathfrak{a} \to B$ such that $g \circ f = h \circ f$. Then,

$$g(\overline{a}) = g \circ f(a) = h \circ f(a) = h(\overline{a})$$

for any $\overline{a} \in A/\mathfrak{a}$. (Actually this is natural since f is surjection.)
Also, let $f : A \to S^{-1}A$ be canonical injection. Let $g, h : S^{-1}A \to B$ such that $g \circ f = h \circ f$. Then,

$$g(a/s) = g(a)g(1/s) = g(a) \cdot g(s)^{-1} = h(a) \cdot h(s)^{-1} = h(a/s).$$

Where $g(s)^{-1} = h(s)^{-1}$ comes from the fact that $g(s/1) = g \circ f(s) = h \circ f(s) = h(s/1)$ and inverse of an element is unique. □

(c) If $U = U'$, then we can make $\rho : A(U) \to A(U')$ and $\nu : A(U') \to A(U)$ satisfying $\phi_f = \nu \circ \phi_g$ and $\phi_g = \rho \circ \phi_f$. (That's what we showed in ii).) Also we showed that ϕ_g, ϕ_f are only depends on U and U' respectively, thus $U = U'$ induces $\phi_g = \phi_f$. Since they are epimorphism, so right cancellable, thus $\rho = 1_{A(U')} = 1_{A(U)} = \nu$.

(d) By ii), we can construct the diagram

$$\begin{tikzcd}
A(U) \arrow[rr, bend left, "\rho"] \arrow[dr, bend right, "\rho'"'] & & A(U'') \\
& A, \arrow[ul, "\phi_U"'] \arrow[ur, "\phi_{U''}"] \arrow[d, "\phi_{U'}"] & \\
& A(U') \arrow[uur, bend right, "\rho''"'] &
\end{tikzcd}$$

commuting each triangle. Now notes that

$$\rho'' \circ \rho' \circ \phi_U = \rho'' \circ \phi_{U'} = \phi_{U''} \text{ and } \rho \circ \phi_U = \phi_{U''}$$

Hence

$$\rho'' \circ \rho' \circ \phi_U = \rho \circ \phi_U$$

and since ϕ_U is epimorphism we showed in ii), it is right cancellable, thus

$$\rho'' \circ \rho' = \rho$$

as desired.

(e) First of all , need to make $I_x = \{U : U \text{ is a basic open set containing } x\}$ be a directed set. Define $X_f \leq X_g$ if $X_g \subseteq X_f$. Then, it is directed set since for any X_f, X_g, then $r((fg)) \subseteq r((f)) \cap r((g))$, hence $V(fg) \supseteq V(f) \cap V(g) \supseteq V(f), V(g)$ implies $X_{fg} \subseteq X_f, X_g$, which implies $X_{fg} \geq X_f, X_g$.
From this relation, we can have $\rho_{fg} : A(X_f) \to A(X_g)$ if $X_f \leq X_g$, i.e., $X_f \supseteq X_g$ using construction of ii). By iii), $\rho_{ff} = 1_{A_f}$ and by iv) $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$ whenever $X_i \leq X_j \leq X_k$. Thus, $\mathbf{M} = (A(X_f), \rho_{fg})$ over the directed set I_x form a direct system. Hence using construction of Exercise 2.14, we have $\varinjlim_{U \in I_x} A(U)$.
Now we want to show that $\varinjlim_{U \in I_x} A(U) \cong A_\mathfrak{p}$, where $\mathfrak{p} = x$ as a prime ideal. To see this, let $\phi_\mathfrak{p} : A \to A_\mathfrak{p}$ be a canonical localization map. Then, for fixed $f \in S = A \setminus \mathfrak{p}$, $\phi_\mathfrak{p}(f)$ is unit. Hence,

by universal property of localization over $S_f = \{1, f, f^2, \cdots\}$, we have a map $\varphi_f : A_f \to A_\mathfrak{p}$ such that

$$\varphi_f \circ \phi_f = \phi_\mathfrak{p}$$

where $\phi_f : A \to A_f$ is canonical localization map. Now we need to show that $\rho_{fg} \circ \varphi_g = \varphi_f$ for $X_f \leq X_g$. We know that triangles in below diagrams commutes, where top triangle commutes from iv).

$$\begin{tikzcd}
A_f \arrow[rr, bend left, "\rho_{fg}"] \arrow[ddr, bend right, "\varphi_f"'] & & A_g \arrow[ddl, bend left, "\varphi_g"] \\
& A \arrow[ul, "\phi_f"] \arrow[ur, "\phi_g"'] \arrow[d, "\phi_\mathfrak{p}"] & \\
& A_\mathfrak{p} &
\end{tikzcd}$$

Thus, we can get

$$\varphi_g \circ \rho_{fg} \circ \phi_f = \phi_\mathfrak{p} = \varphi_f \circ \phi_f.$$

Since ϕ_f is localization map, thus epimorphism. This implies. $\varphi_g \circ \rho_{fg} = \varphi_f$. Hence by Exercise 2.16, there exists a unique homomorphism $\varphi : \varinjlim_{U \in I_x} A(U) \to A_\mathfrak{p}$ such that $\varphi_f = \varphi \circ \rho_f$ where $\rho_f : A_f \to \varinjlim_{U \in I_x} A(U)$ is a canonical injection of direct limit.

Now we need to show that φ is a module isomorphism. To see it is injective, by Exercise 2.15, take an arbitrary element $b \in \varinjlim_{U \in I_x} A(U)$ and its representation $\rho_f(a/f^k)$ such that $\varphi(b) = 0$. Then,

$$0 = \varphi(b) = \varphi \circ \rho_f(a/f^k) = \varphi_f(a/f^k).$$

Notes that $\varphi_f(a/f^n) = a/f^n$. (You can check that this construction is well-defined map and satisfying property of φ_f.) Hence, $a/f^k = 0/1$ in $A_\mathfrak{p}$, there exists $s \in S$ such that $sa = 0$. Then, $\rho_f(a/f^n) = \rho_{sf} \circ \rho_{f,sf}(a/f^n)$. And as we showed in ii) and iv), $\rho_{f,sf}(a/f^n) = as^n/(sf)^n = 0/1$. Thus,

$$b = \rho_f(a/f^k) = \rho_{sf} \circ \rho_{f,sf}(a/f^n) = \rho_{sf}(0/1) = 0.$$

Hence φ is injective.

To see φ is surjective, then pick $a/s \in A_\mathfrak{p}$. Then, $a/s \in A_s$, thus $\rho_s(a/s)$ is in $\varinjlim_{U \in I_x} A(U)$, therefore $\varphi \circ \rho_s(a/s) = \varphi_s(a/s) = a/s$, done.

(f) It is not the exercise in 3.23, but we can extend this construction on any open sets in $X = \operatorname{Spec}(A)$. If U is an open set in X then by definition of base of topology, U is union of X_fs. Say $U = \bigcup_{i \in I} X_{f_i}$. Then, let $S_i = \{1, f_i, f_i^2, \cdots\}$, and $\overline{S}_i$ be saturation of S_i. By Exercise 3.7 ii), $\overline{S}_i = A \setminus \bigcup_{\mathfrak{p} \in X_{f_i}} \mathfrak{p}$. To see this, if a prime ideal $\mathfrak{p}$ doesn't meet S_i, then $f \notin \mathfrak{p}$, thus $\mathfrak{p} \in X_f$. Conversely, if $\mathfrak{p} \in X_f$, then $f \notin \mathfrak{p}$. Thus $f^k \notin \mathfrak{p}$, otherwise $f \in \mathfrak{p}$ by prime property, contradiction. Now just let $X_i := X_{f_i}$. Let $S_U = \bigcap_{i \in I} \overline{S_i}$. Then S_U is saturated multiplicatively closed set since $1 \in S_U$ and each S_is are saturated. Thus,

$$S_U = \bigcap_{i \in I} \left(A \setminus \bigcup_{\mathfrak{p} \in X_{f_i}} \mathfrak{p} \right) = A \setminus \bigcup_{\mathfrak{p} \in X_{f_i}, \forall i \in I} \mathfrak{p} = A \setminus \bigcup_{\mathfrak{p} \in U} \mathfrak{p}.$$

Now define $A(U) := S_U^{-1} A$. For any open subset V of U, the above construction shows that $S_V \supseteq S_U$, thus by Exercise 3.8, we have $\rho_{UV} : A(U) \to A(V)$. If $U = V$ then ρ_{UU} is identity map of $A(U)$ by construction in Exercise 3.8. And for any $U \supseteq V \supseteq W$,

$$\begin{tikzcd}
A(U) \arrow[rr, "\rho_{UW}"] \arrow[dr, "\rho_{UV}"'] & & A(W) \\
& A(V) \arrow[ur, "\rho_{VW}"'] &
\end{tikzcd}$$

commutes since those maps are just identifying map. For example, $a/s \in A(U)$ can be identified as an elements in $A(W)$ by ρ_{UW} or can be identified as an elements in $A(V)$ first then in $A(W)$ lasts, and those are the same. Hence this construction satisfies iii) and iv). Also, this construction is consistent with construction in Exercise 3.23 for the basic open sets. Hence, assignments of ring $A(U)$ for all open sets U of X and the restriction homomorphism ρ_{UV} satisfying condition iii) and iv) are called a presheaf of rings on the open sets. Also the stalk of this presheaf at $x \in X$ is still $A_{\mathfrak{p}}$, since for any $\rho_U(a/s)$, U is open, then U contains a basic open sets, say X_f, thus $\rho_U(a/s) = \rho_f \circ \rho_{U,f}(a/s)$. Hence any elements in the direct limit of open sets can be identified as an elements of direct limit of basic open sets. This implies that their direct limits are the same.

In case of $U = X$, then $S_U = A \setminus \bigcup_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}$. We claim that S_U is a set of all units of A. Notes that if $s \in S_U$ is not a unit, then (s) is proper ideal. Hence it is contained in a maximal ideal, which is in Spec(A), thus $s \notin S_U$, contradiction. Hence $S_U^{-1}A \cong A$, since for any $a/s \in S_U^{-1}$, $a/s = s^{-1}a/1$.

24. Since each U_is are basic open sets, $U_i = X_{f_i}$ for some $f_i \in A$. Also, by Exercise 1.17 v), X is quasi-compact, thus take finite subcover from I, saying that $U_1, \cdots, U_n$ covers X. Then,

$$X = \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} V(f_i)^c = X \setminus \bigcap V(f_i).$$

By Exericse 1.15 iii) $\bigcap V(f_i) = V(\sum(f_i))$ and by Exercise 1.15 i), $V(\sum(f_i)) = \emptyset$ implies $\sum(f_i) = (1)$. Now notes that (f_i) and $\sum_{j \neq i}(f_j)$ are coprime, thus their radical is coprime. Hence from the fact that $(f_i^m) \subseteq r((f_i))$ for any $m \in \mathbb{N}$, $(f_i^m) + \sum(f_j^m)$ are coprime since their radical is (f_i) and $\sum(f_j)$ and use Proposition 1.16. Hence, for any $m \in \mathbb{N}$, we have representation

$$1 = \sum_{i=1}^{n} a_i f_i^m.$$

Now we need to show existence of s. Suppose $s_i \in A_{f_i}$ is given by $s_i = a_i/f_i^{m_i}$ for each i. Then take $m = \max m_i$. Then $s_i = a_i f_i^{m-m_i}/f_i^m$. Denotes $a_i f_i^{m-m_i} = b_i$, so $s_i = b_i/f_i^m$. Now notes that $X_{f_i} \cap X_{f_j} = X_{f_i f_j}$ by Exercise 1.17 i). If $X_{f_i f_j} = \emptyset$, then $f_i f_j$ is nilpotent, thus $A_{f_i f_j} = 0$ since the multiplicative set generated by $f_i f_j$ contains 0, by example 2 in [3][p.38]. In that case, they agrees trivially. So assume that $X_{f_i f_j} \neq \emptyset$. Then, $\rho_{f_i, f_i f_j}(s_i) = b_i f_j^m / f_i^m f_j^m$ and $\rho_{f_j, f_i f_j}(s_j) = b_j f_i^m / f_i^m f_j^m$. For economy of notation, let $g_i = f_i^m$. Then,

$$\rho_{f_i, f_i f_j}(s_i) = b_i g_j / g_i g_j \text{ and } \rho_{f_j, f_i f_j}(s_j) = b_j g_i / g_i g_j.$$

So they agrees if and only if $\exists m_{ij} \in \mathbb{N}$ such that

$$(g_i g_j)^{m_{ij}+1} g_j b_i = (g_i g_j)^{m_{ij}+1} g_i b_j.$$

(Notes that if we can get power of $f_i f_j$, then multiply more so that we can get power of $g_i g_j$ form.) Now let $p = \max_{(i,j) \in [n] \times [n]} m_{ij} + 1$. Then, for any $i, j \in [n]$, we can get always

$$(g_i g_j)^p g_j b_i = (g_i g_j)^p g_i b_j. \tag{1}$$

Then if s exists, then $s/1 = b_i/g_i$ in A_{f_i}. This implies that if such s exists, there exists g_i^k such that $g_i^{k+1} s = g_i^k b_i$. Also, by multiplying g_i arbitrarily many, we can assume that $k \geq p$ for any i. And from the fact that 1 can be linear sum of f_i^l for any $l \in \mathbb{N}$, and $g_i = f_i^m$, so we can say that $1 = \sum_{j=1}^{n} c_j g_i^{m'}$, where $m' \geq p + 1$. Hence

$$g_i^k b_i = \sum_{j=1}^{n} c_j b_i g_i^k g_j^{m'} = \sum_{j=1}^{n} c_j (b_i g_i^p g_j^{p+1}) g_i^{k-p} g_j^{m'-p-1} \underbrace{=}_{\text{Eq. 1}} \sum_{j=1}^{n} c_j (b_j g_j^p g_i^{p+1}) g_i^{k-p} g_j^{m'-p-1} = g_i^{k+1} \sum_{j=1}^{n} c_j b_j g_j^{m'-1}.$$

Thus, let $s = \sum_{i=1}^{n} c_i b_i g_i^p$ where c_i comes from above decomposition of $1 = \sum_{j=1}^{n} c_j g_i^{p+1}$. Then,

$$g_i^p b_i = g_i^{p+1} \sum_{j=1}^{n} c_j b_j g_j^p = g_i^{p+1} s,$$

where first equality comes from the fact that the above equation holds when $k \geq p, m' \geq p+1$. Hence, in A_{f_i},

$$s/1 = g_i^{p+1}s/g_i^{p+1} = g_i^p b_i/g_i^{p+1} = b_i/g_i$$

as desired. Thus $\rho_{X,U_i}(s) = s_i$.

Now, let U_k be an open set in the cover which is not U_i for $i \in [n]$. Then, let $V_i = U_k \cap U_i$. Thus, by assumption, for any $i \in [n]$,

$$\rho_{U_k,V_i}(s_k) = \rho_{U_i,V_i}(s_j) = \rho_{X,V_i}(s) \underbrace{=}_{\text{Exercise 3.23 iv)}} (\rho_{U_k,V_i} \circ \rho_{X,U_k}(s).$$

This shows that $\rho_{U_k,V_i}(s_k - \rho_{X,U_k}(s)) = 0$ for any $i \in [n]$. Notes that since $\{U_i\}_{i\in[n]}$ covers X, thus V_i covers U_k. If we shows the uniqueness of s, then by applying above existence argument and this uniqueness argument on $U_k = \mathrm{Spec}(A_k)$ with finite cover V_is, we can conclude that $s_k = \rho_{X,U_k}(s)$. Thus desired s exists.

Therefore, we need to show that if such s exists, then it is unique. If s, s' are both global section satisfying given conditions, then $\rho_{X,U_i}(s) = s_i = \rho_{X,U_i}(s)$ for any $i \in [n]$. This implies that $\rho_{X,U_i}(s - s') = 0$. Let $t = s - s'$. Then $t \in \ker(\rho_{X,U_i})$ for all $i \in [n]$, thus $t/1 = 0/1$ implies $tf_i^{l_i} = 0$ for some $l_i \in \mathbb{N}$. Take $l = \max_{i\in[n]} l_i$. Then, we have decomposition of 1 by f_i^ls, thus

$$t = t \cdot 1 = \sum c_i f_i^l t = \sum 0 = 0.$$

This shows that $s = s'$. Thus if such s exists globally, then it is unique.

In summary, what we did is that 1) such s exists in any finite subscover, 2) if global section exists, then it is unique, 3) for any basic open set X_k not in subcover, we can make a finite subcover of X_k using the finite subcover for X, and s_k and $\rho_{X,U_k}(s)$ are two global section of $\mathrm{Spec}(A_k)$, so applying 2) on $\mathrm{Spec}(A_k)$ we can get $s_k = \rho_{X,U_k}(s)$. Thus such s exists globally. 4) Now apply 2) to get such s is unique.

25. Let $\mathfrak{p} \in X$ and $k = k(\mathfrak{p})$ be the residue field at $\mathfrak{p}$. By Exercise 21 iv),

$$h^{*-1}(\mathfrak{p}) = \mathrm{Spec}((B \bigotimes_A C) \bigotimes_A k)$$
$$g^{*-1}(\mathfrak{p}) = \mathrm{Spec}(C \bigotimes_A k)$$
$$f^{*-1}(\mathfrak{p}) = \mathrm{Spec}(B \bigotimes_A k)$$

And notes that

$$(B \bigotimes_A C) \bigotimes_A k \underbrace{\cong}_{\text{Exercise 2.15}} B \bigotimes_A (C \bigotimes_A k) \underbrace{\cong}_{\text{Prop 2.14 i)}} B \bigotimes_A (k \bigotimes_A C) \underbrace{\cong}_{\text{Prop 2.14 iv)}} B \bigotimes_A ((k \bigotimes_k k) \bigotimes_A C)$$
$$\underbrace{\cong}_{\text{Exercise 2.15}} B \bigotimes_A (k \bigotimes_k (k \bigotimes_A C)) \underbrace{\cong}_{\text{Exercise 2.15}} (B \bigotimes_A k) \bigotimes_k (k \bigotimes_A C).$$

Hence,

$$\mathfrak{p} \in \mathrm{Im}\, h^* \iff \mathrm{Spec}((B \bigotimes_A k) \bigotimes_k (k \bigotimes_A C)) \neq 0 \iff (B \bigotimes_A k) \bigotimes_k (k \bigotimes_A C) \neq 0$$
$$\underbrace{\iff}_{\text{Exercise 2.3}} (B \bigotimes_A k) \neq 0 \text{ and } (k \bigotimes_A C) \neq 0 \iff \mathrm{Spec}(B \bigotimes_A k) \neq 0 \text{ and } \mathrm{Spec}(k \bigotimes_A C) \neq 0$$
$$\iff g^{*-1}(\mathfrak{p}) \neq 0 \text{ and } f^{*-1}(\mathfrak{p}) \neq 0 \iff \mathfrak{p} \in \mathrm{Im}(f^*) \cap \mathrm{Im}(g^*).$$

Thus,

$$f^*(Y) \cap g^*(Z) = \operatorname{Im}(f^*) \cap \operatorname{Im}(g^*) = \operatorname{Im} h^* = h^*(T).$$

Notes that

Claim XXI. *A commutative ring with unity has zero spectrum if and only if it is zero ring.*

Proof. Every nonzero ring has a maximal ideal by theorem 1.3., which is prime. $\square$

However if a ring doesn't have 1, then it may happen; think $2\mathbb{Z}/8\mathbb{Z} = \{0, 2, 4, 6\}$ mod 8. Then $(4) = \{0, 4\}$ is not a prime since $2 \cdot 6 \equiv 12 \equiv 4$ mod 8. Also $(2) = (6) = \{0, 2, 4, 6\} = 2\mathbb{Z}/8\mathbb{Z}$, thus there is no prime ideal.

26. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then, $f^{*-1}(\mathfrak{p}) \cong \operatorname{Spec}(B \bigotimes_A k(\mathfrak{p}))$ by Exercise 3.21 iv). By Exercise 2.20,

$$B \bigotimes_A k(\mathfrak{p}) = \varinjlim_\alpha (B_\alpha) \bigotimes_A k(\mathfrak{p}) \underbrace{\cong}_{\text{Exercise 2.20}} \varinjlim_\alpha (B_\alpha \bigotimes_A k(\mathfrak{p}))$$

By Exericise 2.21, $\varinjlim_\alpha (B_\alpha \bigotimes_A k(\mathfrak{p}) = 0$ if and only if $B_\alpha \bigotimes_A k(\mathfrak{p}) = 0$ for some α if and only if $f_\alpha^{*-1}(\mathfrak{p}) = \emptyset$ for some α by the above claim. Also the above claim implies that $\varinjlim_\alpha (B_\alpha \bigotimes_A k(\mathfrak{p}) = 0$ if and only if $\operatorname{Spec}(\varinjlim_\alpha (B_\alpha \bigotimes_A k(\mathfrak{p})) = \emptyset$.

Hence, $\mathfrak{p} \in \operatorname{Im} f^*$ if and only if $f^{*-1}(\mathfrak{p}) \neq \emptyset$, if and only if $\alpha, f_\alpha^{*-1}(\mathfrak{p}) \neq \emptyset$ from the contrapositive of the result of above paragraph, which is equivalent to saying that $\mathfrak{p} \in \bigcap_\alpha \operatorname{Im} f_\alpha^*$.

27. (a) Let I be an index set of all B_α. Then let P be collection of all finite subsets of I. We showed that P is directed set. Thus if we let B_J be tensor products of all elements in $\{B_i : i \in J\}$, then $\varinjlim_{J \in P}$ is defined well, by Exercise 2.23.
Then by Exercise 3.26,

$$f^*(\operatorname{Spec}(B)) = \bigcap_{J \in P} f_J^*(\operatorname{Spec}(B_J)).$$

Since each B_J is a finite tensor produtcs, so applying Exercise 3.25 finitely we can get

$$f^*(\operatorname{Spec}(B_J)) = \bigcap_{\alpha \in J} f_\alpha^*(\operatorname{Spec}(B_\alpha)).$$

Hence,

$$f^*(\operatorname{Spec}(B)) = \bigcap_{J \in P} f_J^* f_J^*(\operatorname{Spec}(B_J)) = \bigcap_{J \in P} \bigcap_{\alpha \in J} f_\alpha^*(\operatorname{Spec}(B_\alpha)) = \bigcap_{\alpha \in I} f_\alpha^*(\operatorname{Spec}(B_\alpha)).$$

(b) By Exercise 1.22, $\operatorname{Spec}(B)$ is disjoint union of $\operatorname{Spec}(B_\alpha)$ as an embedded clopen set. Thus each prime ideal of $\operatorname{Spec}(B)$ can be identified as $(\sum_{\alpha \neq \beta} B_\alpha) \oplus \mathfrak{p}_\beta$ for some $\mathfrak{p}_{beta} \in \operatorname{Spec}(B_\beta)$. Hence,

$$f^*((\sum_{\alpha \neq \beta} B_\alpha) \oplus \mathfrak{p}_\beta) = f^{-1}((\sum_{\alpha \neq \beta} B_\alpha) \oplus \mathfrak{p}_\beta) = \{a \in A : f_\beta(a) \in \mathfrak{p}_\beta\} = f_\alpha^*(\mathfrak{p}_\beta).$$

Since β was arbitrarily chosen, an from the disjoint union,

$$f^*(\operatorname{Spec}(B)) = \bigcup_\beta f^*(\{(\sum_{\alpha \neq \beta} B_\alpha) \oplus \mathfrak{p}_\beta : \mathfrak{p}_\beta \in \operatorname{Spec}(B_\beta)\}) = \bigcup_\beta f_\beta^*(\operatorname{Spec}(B_\beta),$$

as desired.

(c) Let $\tau := \{C_f : f : A \to B \text{ for any ring B}\}$ where $C_f := f^*(\mathrm{Spec}(B))$. Then, intersection of C_fs are also closed, since we can construct tensor algebra whose $f^*(\mathrm{Spec}(B))$ is the given intersection, by i). Similarly, for any finite union of C_f, we can construct a direct product whose $f^*(\mathrm{Spec}(B))$ is the given finite union. Thus, it suffices to show that $\emptyset$ and $X = \mathrm{Spec}(A)$ is in τ. Since $C_{1_A} = X$ and $C_{A\to 0} = \emptyset$, we are done.

To see that it is finer than Zariski topology, recall that $V(\mathfrak{a}) \cong \mathrm{Spec}(A/\mathfrak{a})$ by Exercise 1.21 with a surjective ring homomorphism $f : A \to A/\alpha$, whose kernel is $\mathfrak{a}$. Thus $V(\mathfrak{a}) = f^*(\mathrm{Spec}(A/\mathfrak{a})) \in \tau$. Notes that it is not always finer; for example, if A has only one prime ideal, then $\mathrm{Spec}(A) = \{\bullet\}$, thus any image of f^* is either $\mathrm{Spec}(A)$ or $\emptyset$.

(d) One of the version of definition of quasi compactness is that every collection of closed sets of X with empty intersection has some finite subsets of the collection whose intersection is empty.

Suppose that $\{C_f : f : A \to B_f\}$ be a subset of τ whose intersection is empty. And by ii), their intersection is corresponding to tensor algebras of B_f, say B. Let $g : A \to B$. Then,

$$\emptyset = g^*(\mathrm{Spec}(B)).$$

This implies that $\mathrm{Spec}(B) = \emptyset$, thus by above claim, $B = 0$. Then, by Exercise 2.21, there exists B_J where J is finite subsets of B_f, such that $B_J = 0$. Thus,

$$\emptyset = f^*(B_J) = \bigcap_{\alpha\in J} f_\alpha^*(\mathrm{Spec}(B_\alpha))$$

by i).

28. (a) X_g is set of prime ideals not meeting $S = \{1, g, g^2, \cdots\}$. Hence, by Proposition 3.11, $X_g = f^*(\mathrm{Spec}(S^{-1}A))$. Hence it is closed. It is open since $X_g = V(g)^c$, where $V(g)$ is still closed in this topology.

(b) Let $\mathfrak{p} \neq \mathfrak{q}$. Then, there exists $f \in \mathfrak{p} \setminus \mathfrak{q}$. Thus, X_f contains $\mathfrak{q}$ but not $\mathfrak{p}$. Since X_f is clopen, its complement is clopen, which implies X_f^c contains $\mathfrak{p}$ but not $\mathfrak{q}$. Thus there is two disjoint open sets separating $\mathfrak{p}$ and $\mathfrak{q}$.

(c) Since this map is identity, it is bijective. Also, by definition of subbase, $X_{C'}$ is generated by X_g and $X \setminus X_g$, thus any closed sets in $X_{C'}$ is finite union of arbitrary intersection of these X_g and $X \setminus X_g$s. By i), these subbase is still clopen in C, the constructible topology. Thus every closed sets in $X_{C'}$ is still finite union of arbitrary intersection of closed sets, thus closed in X_C. Hence it is continuous. Finally, we know that $X_{C'}$ is Hausdorff by ii), and X_C is quasi compact by Exercise 3.27 iv). And it is well-known that a continuous bijection from compact domain to Hausdorff codomain is homeomorphism.

(d) Thus X_C is compact Hausdorff by homeomorphism in iii). It suffices to see that it is totally disconnects. As we can see in the proof of ii), for any $\mathfrak{p} \neq \mathfrak{q}$, there exists $\mathfrak{p} \in X_f$ and $\mathfrak{q} \in X \setminus X_f$. Since both X_f and X_f are clopen, so they are disjoint. Since we choose $\mathfrak{q}$ arbitrarily, only connected component containing $\mathfrak{p}$ is singleton $\{\mathfrak{p}\}$. Since $\mathfrak{p}$ was chosen arbitrarily, done.

29. Any closed set in the constructible topology of $\mathrm{Spec}(B)$ can be denoted as $g^*(\mathrm{Spec}(C))$ for some ring homomorphism $g : B \to C$. Thus,

$$f^*(g^*(\mathrm{Spec}(C)) = (g \circ f)^{ast}(\mathrm{Spec}(C))$$

by Exercise 1.21 vi). Since $g \circ f : A \to C$ is also a ring homomorphism, so $(g \circ f)^{ast}(\mathrm{Spec}(C))$ is closed set in the constructible topology of $\mathrm{Spec}(A)$.

30. Let $1_X : X_C \to X$ where X_C is X with constructible topology and X is just with Zariski topology. Since every closed set in Zariski topology is also closed in X_C, so 1_X is continuous bijection. If $A/\mathfrak{R}$ is abolutely flat, then by Exercise 3.11 iv), X is Hausdorff, thus by the fact in topology stating that a continuous bijection from compact domain to Hausdorff codomain is homeomorphism, 1_X is homeomorphism.

Conversely, if 1_X is homeomorphism, then X is Hausdorff by Exercise 3.28 iv). Hence by Exercise 3.11, $A/\mathfrak{R}$ is absolutely flat.

4 Primary Decomposition

Σ is an ***isolated set*** of prime ideals belonging to $\mathfrak{a}$ if it satisfies the following condition: if $\mathfrak{p}'$ is a prime ideal belonging to $\mathfrak{a}$ and $\mathfrak{p}' \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Sigma$, then $\mathfrak{p}' \in \sigma$.

Theorem 4.10 (2nd uniqueness theorem). *Let $\mathfrak{a}$ be a decomposable ideal, let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{a}$, and let $\{\mathfrak{p}_{i_1}, \cdots, \mathfrak{p}_{i_m}\}$ be an isolated set of prime ideals of $\mathfrak{a}$. Then, $q_{i_1} \cap \cdots q_{i_m}$ is independent of the decomposition.*

Proof. Notes that any associate prime of $\mathfrak{a}$ is uniquely determined by $\mathfrak{a}$, due to the 1st uniqueness theorem (Theorem 4.5). Now let $S = A - \mathfrak{p}_{i_1} \cup \cdots \cup \mathfrak{p}_{i_m}$. Thus, $S(\mathfrak{a})$ is also uniquely determined by $\mathfrak{a}$ Now, suppose that there is another minimal primary decomposition $\mathfrak{a} = \bigcap_i \mathfrak{q}'_i$. Then, we can assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_{m'}$ meets S, and the other associate primes over this decomposition doesn't meet S. Then Proposition 4.9 shows that

$$S(\mathfrak{a}) = \bigcap_{i=1}^{m'} \mathfrak{q}'_i$$

Since $S(\mathfrak{a})$ is contraction of $S^{-1}\mathfrak{a}$, thus already determined. This implies that

$$\bigcap_{j=1}^{m} \mathfrak{q}_{i_j} = \bigcap_{j=1}^{m'} \mathfrak{q}'_i.$$

Hence the intersection is independent of decomposition. □

Corollary T. *he isolated primary components (i.e., the primary components $\mathfrak{q}_i$ corresponding to minimal prime ideals $\mathfrak{p}_i$) are uniquely determined by $\mathfrak{a}$.*

Proof. Let $\Sigma = \{\mathfrak{p}\}$, a minimal associate prime of $\mathfrak{a}$ and apply theorem 4.8 □

Proposition 4.12$*$, [4]**.** *Let A be a ring, S a multiplicatively closed subset of A. Write $\phi_S : A \to S^{-1}A$ a canonical morphism. For any ideal $\mathfrak{a}$, let $S(\mathfrak{a})$ denote the contraction along ϕ_S of $S^{-1}\mathfrak{a}$. The ideal $S(\mathfrak{a})$ is called the saturation of $\mathfrak{a}$ with respect to S.*

1. $\bigcup_{s \in S} (\mathfrak{a} : s) = \{x \in A : \exists s \in S \text{ s.t. } sx \in \mathfrak{a}\} = S(\mathfrak{a}) = \mathfrak{a}^{ec} \supseteq \mathfrak{a}$.
2. $S(0) = \ker(\phi_S)$.
3. *Let $S_\mathfrak{p} = A \setminus \mathfrak{p}$ for $\mathfrak{p}$ a prime ideal of A. If $\mathfrak{q}$ is $\mathfrak{p}$-primary, then $S_\mathfrak{q}(\mathfrak{q}) = \mathfrak{q}$.*
4. *$S_\mathfrak{p}(0)$ is contained in every $\mathfrak{p}$-primary ideal of A.*
5. *If $S_1 \subseteq S_2$ are multiplicative set of A, then $S_1(\mathfrak{a}) \subseteq S_2(\mathfrak{a})$.*
6. *If $\mathfrak{b}$ is an ideal of A containing $\mathfrak{a}$, then $S(\mathfrak{a}) \subseteq S(\mathfrak{b})$.*

Proof. For i), First equality is clear by definition. Second equality comes from the fact that

$$x \in \{x \in A : \exists s \in S \text{ s.t. } sx \in \mathfrak{a}\} \iff x/1 = a/s \text{ for some } a \in \mathfrak{a}, s \in S \iff x/1 \in \mathfrak{a}^e \iff x \in \mathfrak{a}^{ec}.$$

The third equality comes from definition, and last inclusion comes from Proposition 1.17.

For ii), $(0)^e = (0)$. Hence $(0)^{ec} = \{x \in A : \phi_S(x) = 0\} = \ker \phi_S$.

For iii), Use Proposition 4.9 with $\mathfrak{a} = \mathfrak{q}$, thus S doesn't meet $\mathfrak{p}$ implies $S(\mathfrak{a}) = \mathfrak{q}$.

For iv), for any $x \in S_\mathfrak{p}(0)$, $x/1 = 0/1$ implies $sx = 0$ for some $s \in S_\mathfrak{p}$, thus $sx = 0 \in \mathfrak{q}$ for any $\mathfrak{p}$-primary ideal $\mathfrak{q}$. Since $s \notin \mathfrak{p}$, $x \in \mathfrak{q}$.

For v), if $x \in S_1(\mathfrak{a})$, then $\exists s \in S_1 \subseteq S_2$ such that $sx \in \mathfrak{a}$, thus $x \in S_2(\mathfrak{a})$ by definition.

For vi),

$$\mathfrak{a} \subseteq \mathfrak{b} \implies \mathfrak{a}^e \subseteq \mathfrak{b}^e \implies \mathfrak{a}^{ec} \subseteq \mathfrak{b}^{ec}$$

since below claim. □

Claim XXII. *Both extension and contaction preserves order of inclusion.*

Proof. In case of extension, it is clear since generators of $\mathfrak{b}^e$ contains generators of $\mathfrak{a}^e$. In case of contraction, it is also clear since it is just preimage. $\square$

1. Let $\mathfrak{a}$ has a minimal primary decomposition $\mathfrak{a} = \bigcap_{i=1}^n q_i$. In p.52 [3], minimal (isolated) primes of $\mathfrak{a}$ must be a subset of all minimal elements of $\{\mathfrak{p}_i\}_{i=1}^n$ where $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Thus, a set of all minimal associate primes of $\mathfrak{a}$ is finite set, since the decomposition is finite.

   In other hands, by Exercise 1.20 iv), the irreducible components of $\mathrm{Spec}(A/\mathfrak{a})$ is the closed sets $V(\overline{\mathfrak{p}})$ where $\overline{\mathfrak{p}}$ is minimal primes of $A/\mathfrak{a}$. By Proposition 1.1, those are image of minimal prime ideal $\mathfrak{p}$ with respect to set of all prime ideals containing A. By Proposition 4.6, the set of all minimal prime ideals containing $\mathfrak{a}$ is the same as the set of all minimal associate primes of $\mathfrak{a}$, which is finite. Hence, $\mathrm{Spec}(A/\mathfrak{a})$ has also finitely many irreducible components.

2. Suppose that $\mathfrak{a}$ is decomposable. (Otherwise, this statement is vacuously true.) Then, $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$. Now the 1st uniqueness theorem states that for any minimal primary decomposition of $\mathfrak{a}$, say $\mathfrak{a} = \bigcap_{i=1}^k \mathfrak{q}_i'$, its associated primes of $\mathfrak{a}$ are the same as associate primes given by $\bigcap_{i=1}^n \mathfrak{q}_i$. In other words,

   $$\{\mathfrak{p}_i\}_{i=1}^n = \{\mathfrak{p}_i\}_{i=1}^k.$$

   This implies $k = n$, hence any minimal decomposition of $\mathfrak{a}$ has the same number of primary components. Thus,

   $$I = r(I) = r(\bigcap_{i=1}^n \mathfrak{q}_i) = \bigcap_{i=1}^n r(q_i) = \bigcap_{i=1}^n \mathfrak{p}_i.$$

   Thus, if there is embedded prime, say $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$, then $I = \bigcap_{i=2}^n \mathfrak{p}_i$ is another minimal primary decomposition consisting of $n-1$ primary ideals, which contradicting the 1st uniqueness theorem.

3. Let $\mathfrak{q}$ be a primary ideal of A. Then since A is absolutely flat, $A/\mathfrak{q}$ is also absolutely flat by Exercise 2.28. By definition of primary ideal, every zero divisor of $A/\mathfrak{q}$ is nilpotent. Also Exercise 2.28 states that every non unit in A is a zero divisor. Hence, for any $x \in A/\mathfrak{q}$, x is unit or nilpotent. By Exercise 1.10, $A/\mathfrak{q}$ has only one prime ideal. Therefore, $A/\mathfrak{q}$ is local ring. Then, by Exercise 2.28 stating that local absolutely flat ring is field, $A/\mathfrak{q}$ is field. Hence, $\mathfrak{q}$ is maximal ideal.

4. First of all $(2,t)$ is maximal, since $f : \mathbb{Z}[t] \to \mathbb{Z}/2\mathbb{Z}$ by $f(t) \mapsto f(0) \bmod 2$ is surjective homomorphism and $(2,t) = \ker f$. However, $g : \mathbb{Z}[t] \to \mathbb{Z}/4\mathbb{Z}$ by $f(t) \mapsto f(0) \bmod 4$, is surjective map and $(4,t) = \ker g$. Thus, $\mathbb{Z}[t]/(4,t) \cong \mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/4\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$. Notes that $\overline{1}, \overline{3}$ are unit, and $\overline{2}, \overline{0}$ are nilpotent and zero divisors. Thus, $(4,t)$ is primary. To see it is $\mathfrak{m}$-primary, $r(4,t) = (2,t)$. (Definitley, $2 \in r(4,t)$, thus $r(4,t) \supseteq (2,t)$ and $(2,t)$ is maximal implies the other direction of inclusion.) Notes that $(2,t)^2 = (4, t^2, 2t) \subsetneq (4,t)$ implies that $(4,t)$ is not power of $\mathfrak{m}$.

5. First of all, $\mathfrak{a} = (x^2.xz, xy, yz)$. Since $x^2, xz, xy, yz \in \mathfrak{m}^2$, $\mathfrak{a} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Conversely, if $f \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$, then since $f \in \mathfrak{m}^2$,

   $$f = ax^2 + by^2 + cz^2 + dxy + exz + gyz$$

   for some $a, b, c, d, e, g \in K[x, y, z]$. Since $f \in \mathfrak{p}_1$, c should divisible by x or y, thus we can rewrite it as

   $$f = ax^2 + by^2 + dxy + exz + gyz$$

   Also, since $f \in \mathfrak{p}_2$ b should be divisible by x or z. Thus, we can rewrite it as

   $$f = ax^2 + dxy + exz + gyz$$

   Thus $f \in \mathfrak{a}$, this shows the equality. To see it is reduced, notes that

   $$\mathfrak{p}_1 \cap \mathfrak{m}^2 = (x^2, y^2, xz, yz, xy)$$
   $$\mathfrak{p}_2 \cap \mathfrak{m}^2 = (x^2, z^2, xz, yz, xy)$$
   $$\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x, yz)$$

are not the same as $\mathfrak{a}$. Now notes that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are contained in $\mathfrak{m}$. Hence $\mathfrak{m}$ is embedded prime. However, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ doesn't have inclusion relationship. Thus, those are two minimal associated primes of $\mathfrak{a}$

6. From Exercise 1.16 i), every maximal ideal of $C(X)$ is of forms $\mathfrak{m}_x = \{f \in C(X) : f(x) = 0\}$ for some $x \in X$. We claim that every primary ideal is contained in a unique maximal ideal. To see this, let $\mathfrak{a} \subseteq \mathfrak{m}_x \cap \mathfrak{m}_y$. Since X is Hausdorff, there exists disjoint open neighborhoods of x and y, say U_x and U_y. Since X is compact Hausdorff, Urysohn's lemma implies that there exists a continuous function $f_x, f_y : X \to \mathbb{R}$ such that $f_x(x) = 1$, $f_x(X \setminus U_x) = 0$ and $f_y(X \setminus U_y) = 0, f_y(y) = 1$. Hence, $fg = 0 \in \mathfrak{a}$. Since $\mathfrak{a}$ is primary, $f \in \mathfrak{a}$ or $g^n \in \mathfrak{a}$. However, neither is in $\mathfrak{m}_x \cap \mathfrak{m}_g \supseteq \mathfrak{a}$ which implies $\mathfrak{a}$ is not primary.

   Now notes that X is infinite space, there are infinitely many maximal ideals, therefore there are infinitely many minimal prime ideals. Now suppose that (0) is decomposable, say $(0) = \bigcap_{i=1}^n \mathfrak{q}_i$. Then, by Proposition 4.6, $\{\mathfrak{p}_i\}_{i=1}^n$ contains a set of minimal prime ideals belonging to (0), and which is the same as a set of minimal elements of in the set of prime ideals containing (0), Since every prime ideal contains (0), this implies that a set of all minimal prime ideals of $C(X)$ is finite. Which contradicts the fact that there are infinitely many minimal prime ideals.

7. (a) Notes that $\mathfrak{a}[x] \subseteq \mathfrak{a}^e$, since any polynomial in $\mathfrak{a}[x]$ is linear combination of elements in $\mathfrak{a}$. Conversely, any elements in $\mathfrak{a}^e$ can be denoted as $\sum_{i=1}^n a_i f_i$ where $a_i \in \mathfrak{a}$ and $f_i \in A[x]$. Since $a_i f_i$ is a polynomial whose coefficients are in $\mathfrak{a}$, thus $\sum_{i=1}^n a_i f_i \in \mathfrak{a}[x]$.

   (b) Let $\phi : A[x] \to A/\mathfrak{p}[x]$ by $ax^i \mapsto \bar{a}x^i$. Then, $\ker \phi$ contains $\mathfrak{p}[x]$ since every coefficient is mapped into zero. Conversely, if $f \in \ker \phi$, then each coefficient of f should lie in $\mathfrak{p}$. Thus $f \in \mathfrak{p}[x]$. Since it is surjective morphism, $A[x]/\mathfrak{p}[x] \cong A/\mathfrak{p}[x]$. Since $A/\mathfrak{p}$ is integral domain, so does $A/\mathfrak{p}[x]$ by Exercise 1.2 iii). (If f is zero divisor in $A/\mathfrak{p}[x]$, then by Exercise 1.2 iii), there exists $a \in A/\mathfrak{p}$ such that $af = 0$. If f has degree n, then its n-th coefficient a_n is nonzero, thus aa_n is also nonzero since $A/\mathfrak{p}$ has no zero divisor. Thus, af also has degree n. Thus $af = 0$ implies $f = 0$.) Hence $\mathfrak{p}[x]$ is prime ideal in $A[x]$.

   (c) By the same reasoning, we know that $A[x]/\mathfrak{q}[x] \cong A/\mathfrak{q}[x]$. Thus to see that $\mathfrak{q}[x]$ is primary, we need to show that $A/\mathfrak{q}[x] \neq 0$ (which is already shown) and $A/\mathfrak{q}[x]$ every zero divisor in $A/\mathfrak{q}[x]$ is nilpotent. Let $f \in A/\mathfrak{q}[x]$ be a zero divisor. By Exercise 1.2 iii), $\exists a \neq 0 \in A$ such that $af = 0$. This implies that a is a zero divisor of $A/\mathfrak{q}$ since $af = 0$ implies $aa_n = 0$ where a_n is the leading coefficient of f. Thus a and a_n are nilpotent in $A/\mathfrak{q}$. Similarly, if $a_i \neq 0$ for i-th coefficient of f, then $aa_i = 0$ implies that a_i are nilpotent in $A/\mathfrak{q}$. Thus, all coefficients of f are nilpotent. By Exercise 1.2 ii), f is nilpotent. This shows that $\mathfrak{q}[x]$ is primary ideal. Now it suffices to show that $r(\mathfrak{q}[x]) = \mathfrak{p}[x]$. To see this, let $f = \sum_{i=1}^n a_i x^i \in \mathfrak{p}[x]$. Then, for each a_i, $\exists n_i$ such that $a_i^{n_i} \in \mathfrak{q}$. Thus, $f^{\sum_{i=1}^n n_i + 1}$ consists of all terms with a form

$$\binom{\sum_{i=1}^n n_i + 1}{k_1, \cdots, k_n} \prod_{j=1}^n a_j^{k_i} x^{\sum_{j=1}^n k_j}.$$

   By pigeonhole principle, there exists at least one k_i which is greater than n_i. Thus the coefficient is in $\mathfrak{q}_i$, thus $\mathfrak{p}[x] \subseteq r(\mathfrak{q}[x])$. Conversely, suppose $f \in r(\mathfrak{q}[x])$. If f is of degree 0, then $f \in \mathfrak{p} \subseteq \mathfrak{p}[x]$. Suppose it holds for degree p. Then, if f is of degree $p+1$, then from $f^n \in \mathfrak{q}[x]$ for some n, $(a_{p+1})^n \in \mathfrak{q}$, thus $a_{p+1} \in r(\mathfrak{q}) = \mathfrak{p}$, and $a_{p+1}x^{p+1} \in r(\mathfrak{q}[x])$. Hence, $f - a_{p+1}x^{p+1} \in r(\mathfrak{q})$. By inductive hypothesis, $f - a_{p+1}x^{p+1} \in \mathfrak{p}[x]$. Since $a_{p+1}x^{p+1} \in \mathfrak{p}[x]$, so does f. Done.

   (d) First of all, to see $\mathfrak{a}[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$, let $f \in \mathfrak{a}[x]$. It is equivalent to say that all coefficients of f is in $\bigcap_{i=1}^n \mathfrak{q}_i$. This is equivalent to say that $f \in \mathfrak{q}_i[x]$ for all i, done.
   By iii), $\mathfrak{a}[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$ is primary decomposition. To see it is minimal, delete $q_j[x]$. Then, $\bigcap_{i\neq j}^n \mathfrak{q}_i[x] = \left(\bigcap_{i\neq j}^n \mathfrak{q}_i\right)[x]$ by the same reasoning as we did for previous paragraph. Then, $\bigcap_{i\neq j}^n \mathfrak{q}_i \neq \mathfrak{a}$ since $\bigcap_{i=1}^n \mathfrak{q}_i$ is minimal primary decomposition. Thus, $\bigcap_{i=1}^n \mathfrak{q}_i[x]$ is also minimal primary decomposition.

   (e) If there exists a prime ideal $\mathfrak{q}$ such that $\mathfrak{a}[x] \subseteq \mathfrak{q} \subseteq \mathfrak{p}[x]$, then its contraction gives us

$$\mathfrak{a} \subseteq \mathfrak{q} \cap A \subseteq \mathfrak{p}$$

with $A \cap \mathfrak{q}$ prime, by p.9 [3]. By minimality of $\mathfrak{p}$, $A \cap \mathfrak{q} = \mathfrak{p}$. Thus, $\mathfrak{q}$ contains $\mathfrak{p}$, hence $\mathfrak{q}$ contains $\mathfrak{p}^e = \mathfrak{p}[x]$. This shows $\mathfrak{q} = \mathfrak{p}[x]$. Hence, $\mathfrak{p}[x]$ is minimal prime ideal of $\mathfrak{a}[x]$.

8. $A/\mathfrak{p}_i \cong k[x_{i+1}, \cdots, x_n]$ implies that $\mathfrak{p}_i$ is prime since polynomial ring over field is integral domain. To see its power is primary, let $\mathfrak{q} = \mathfrak{p}_i^m \cap k[x_1, \cdots, x_i]$ for fixed $m \in \mathbb{N}$, let $\mathfrak{p} = \mathfrak{p}_i \cap k[x_1, \cdots, x_i]$.Then, $\mathfrak{p}$ is maximal in a similar way we've seen in Exercise 4.5 for (x, y, z). Also, $\mathfrak{p}[x_{i+1}, \cdots, x_n] = \mathfrak{p}_i$ since $\mathfrak{p} \subseteq \mathfrak{p}_i$ implies $\mathfrak{p}[x_{i+1}, \cdots, x_n] \subseteq \mathfrak{p}$ and for any $f \in \mathfrak{p}$, $f = \sum_{j=1}^{i} f_j x_j$ for some $f_j \in k[x_1, \cdots, x_n]$, hence all coefficients of $f_j x_j$ as a polynomial of $x_{i+1}, \cdots, x_n$ over $k[x_1, \cdots, x_i]$ contained in $x_j k[x_1, \cdots, x_i]$, hence is in $\mathfrak{p}[x_{i+1}, \cdots, x_n]$. Also, $\mathfrak{q}[x_{i+1}, \cdots, x_n] = \mathfrak{p}_i^m$ by the similar way.

   If $\mathfrak{q}$ is $\mathfrak{p}$-primary, then $\mathfrak{q}[x_{i+1}, \cdots, x_n] = \mathfrak{p}_i^m$ is $\mathfrak{p}[x_{i+1}, \cdots, x_n] = \mathfrak{p}$ primary by applying Exercise 4.7 iii) $n - i$ times. Thus it suffices to show that $\mathfrak{q}$ is $\mathfrak{p}$-primary. Notes that $\mathfrak{q} = \mathfrak{p}^m$ in $k[x_1, \cdots, x_i]$. Thus, it suffices to show that $k[x_1, \cdots, x_i]/\mathfrak{q} = k[x_1, \cdots, x_i]/\mathfrak{p}^m \neq 0$ and all zero divisors are nilpotent. Suppose that $f \in k[x_1, \cdots, x_i]/\mathfrak{q}$ is zero divisor. Then, $\exists g$ such that $fg = 0$. If f has nonzero constant, say $f = f_0 + c$ where f_0 is a polynomial with zero constant, then $0 = fg = f_0 g + cg$. Since $cg \neq 0$, $f_0 g = -cg$. However, degree of $f_0 g$ is zero or greater than that of cg since degree of f is greater than 1. This implies $\deg f_0 g = 0$, then $g = 0$, contradiction. (The other case cannot happen since $f_0 g$ and cg have the same degree.) Thus, f should have zero constant. And as we know, if we multiply f m times, then every term in f should have degree m, which is zero. Thu,s f is nilpotent.

9. Suppose x is zero divisor. Then there exists $a \neq 0 \in A$ such that $xa = 0$. Thus, $x \in (0 : a)$, and since $a \neq 0$, $(0 : a) \neq (1)$. Hence, apply Exercise 1.8 on a ring $A/(0 : a)$ to get an existence of a minimal prime ideal $\overline{\mathfrak{p}}$ on $A/(0 : a)$, and by Proposition 1.1 their contraction $\mathfrak{p}$ is minimal prime ideal containing $(0; a)$. Thus there exists $\mathfrak{p} \in D(A)$ such that $x \in \mathfrak{p}$.

   Conversely, if $x \in \mathfrak{p} \in D(A)$, then $\mathfrak{p}$ is a minimal prime ideal containing $(0 : a)$ for some $a \neq 0$. If $x \in (0 : a)$, done. Otherwise, notes that by Proposition 1.1, $\overline{\mathfrak{p}}$ on $A/(0 : a)$ is still minimal prime ideal containing 0. Let $\overline{S} = A/(0 : a) \setminus \overline{\mathfrak{p}}$. By Exercise 3.6, $\overline{S}$ is a maximal multiplicative closed subset of $A/(0 : a)$ such that $\overline{0} \notin \overline{S}$. Now since $\overline{x} \neq 0$, let $S' = \{s\overline{x}^n : s \in \overline{S}, n \in \mathbb{N}\}$. First of all, S' is multiplicatively closed, since any two elements $s\overline{x}^m$ and $s'\overline{x}^l$, there product is $ss'\overline{x}^{m+l} \in S'$. Also, $S' \neq \overline{S}$ since $\overline{x} \in \mathfrak{p} = A/(0 : a) \setminus S$ but $\overline{x} \in S'$. Thus, S' has $\overline{0}$, which implies $s\overline{x}^m = \overline{0}$ for some $m \in \mathbb{N}$ and $s \in S$. This implies $sx^m \in (0 : a)$, thus $sx^m a = 0$. This implies x is zero divisor in A.

   For the second statement, notes that $\operatorname{Spec}(S^{-1}A)$ as a subset of $\operatorname{Spec}(A)$ is

$$\{x \in \operatorname{Spec}(A) : \mathfrak{p}_x \cap S = \emptyset\}.$$

   Now observe that with respect to a canonical map $\phi : A \to S^{-1}A$,

$$\operatorname{Ann}(x)^e = S^{-1} \operatorname{Ann}(x) = S^{-1}(0 : x) = (S^{-1}0 : S^{-1}x) = (0/1 : x/1) = \operatorname{Ann}(x/1)$$

   by Corollary 3.15. Also notes that $\operatorname{Ann}(x/1) = \operatorname{Ann}(x/s)$ since if $a/t \in \operatorname{Ann}(x/1)$, then $ax/t = 0$ implies $\exists q \in S$ such that $axq = 0$. Then, ax/ts is also zero since this q makes $q(ax - ts0) = qax = 0$. Converse is the same.

   By Proposition 3.11, every ideal in $S^{-1}A$ is an extended ideal. And by Proposition 3.11 iv), let $S^{-1}\mathfrak{p}$ be an arbitrary prime ideal in $S^{-1}A$ induced by $\mathfrak{p} \in \operatorname{Spec}(A)$. If $S^{-1}\mathfrak{p} \in D(S^{-1}A)$, then it is a minimal prime ideal of $(0 : a/s) = \operatorname{Ann}(a/s) = \operatorname{Ann}(a/1)$. Thus, by 1-1 correspondence (Proposition 3.11 iv)), $\mathfrak{p}$ is minimal prime ideal in A containing $(0 : a)$. (First of all, for any $x \in (0 : a)$, $x/1 \in (0 : a/1)$, hence $x/1 \in S^{-1}\mathfrak{p}$, thus its preimage x is in $\mathfrak{p}$. And if there is another prime ideal $\mathfrak{q}$ conatining $(0 : a)$ but contained in $\mathfrak{p}$, then $S^{-1}\mathfrak{q}$ contains $\operatorname{Ann}(a/1)$ but contained in $\mathfrak{p}$ because of inclusion relationship of its generating set. It contradicts the fact that $S^{-1}\mathfrak{p}$ is minimal prime ideal containing $\operatorname{Ann}(a/1)$. This shows that if $S^{-1}\mathfrak{p} \subseteq D(S^{-1}A)$ then $\mathfrak{p} \in D(A)$. Since $\mathfrak{p}$ is in the image of $D(S^{-1}A)$ if and only if $S^{-1}\mathfrak{p} \subseteq D(S^{-1}A)$, this implies $D(S^{-1}A) \subseteq D(A) \cap \operatorname{Spec}(S^{-1}A)$. Conversely, if $\mathfrak{p}$ is a minimal ideal containing $(0 : a)$. Then, $S^{-1}\mathfrak{p}$ contains $\operatorname{Ann}(a/1)$ and minimal by the same argument, and 1-1 correspondence. This implies $D(S^{-1}A) \supseteq D(A) \cap \operatorname{Spec}(S^{-1}A)$.

   If zero ideal has primary decomposition, the by the 1st uniqueness theorem, there are finite prime ideals of form $r(0 : x)$ for some $x \in Z \subseteq A$ with $|Z| < \infty$. Since all prime ideals are radical, a prime ideal containing $(0 : x)$ should contain $r(0 : x)$. This shows that all associate primes of zero ideal should be contained in $D(A)$. Conversely, suppose $\mathfrak{p}$ is a minimal prime ideal containing $(0 : a)$ for some $a \neq 0 \in A$. Suppose $\{\mathfrak{p}_j\}_{j=1}^n$ be set of all associate primes of 0. Since $\mathfrak{p}$ contain 0, by Proposition 4.6, it contains $\mathfrak{p}_j$ for some j, which is minimal associate prime. If $a \notin \bigcup_{i=1}^n \mathfrak{p}_i$, where $\mathfrak{p}_i$ is an associate primes of 0 (and n is the number of all distinct associate prime ideals of 0), then $(0 : a) = 0$. Minimality of $\mathfrak{p}$ with Proposition 4.6 implies that $\mathfrak{p} = \mathfrak{p}_j$.

If $a \in \mathfrak{p}_i$ for some $i \in [n]$, then by Proposition 4.7, $(0 : a) \neq 0$. Since every elements in $(0 : a)$ is zero divisor, $(0 : a) \subseteq \bigcup_{i=1}^n \mathfrak{p}_i$. By Proposition 1.11 i), $(0 : a) \subseteq \mathfrak{p}_j$ for some j. Since $\mathfrak{p}$ is minimal prime ideal conatining $(0 : a)$,

$$\mathfrak{p}_j \supseteq \mathfrak{p}.$$

If $\mathfrak{p}$ is minimal prime ideal containing 0, then it is an associate prime of 0, done. Otherwise, there is a minimal prime ideal $\mathfrak{p}_k$ contained in $\mathfrak{p}$. Thus,

$$\mathfrak{p}_j \supseteq \mathfrak{p} \supseteq \mathfrak{p}_k.$$

Now for given minimal decomposition $0 = \bigcap_{i=1}^n q_i$, think

$$\mathfrak{q} = (\bigcap_{i \neq j}^n q_i) \cap \mathfrak{q}' \text{ where } \mathfrak{q}' = \mathfrak{p} \cap \mathfrak{q}_i.$$

First of all, if $\mathfrak{p} = r(\mathfrak{q}_i)$ for some i, then done. Thus assume that $\mathfrak{p} \neq r(\mathfrak{q}_i)$ for any i. Now think about second condition of minimal prime decomposition. First of all, $\mathfrak{q}' \not\supseteq \bigcap_{j \neq i} q_j$ since $\mathfrak{q}_i$ doesn't. For any $l \neq i$, $\bigcap_{q \neq l} \mathfrak{q}_q \not\subseteq \mathfrak{q}_l$ from the original minimal primary decomposition,

$$\mathfrak{p} \cap (\bigcap_{q \neq l} \mathfrak{q}_q) \not\subseteq \mathfrak{q}_l \cap \mathfrak{q}.$$

Since $\mathfrak{p} \cap (\bigcap_{q \neq l} \mathfrak{q}_q) = (\bigcap_{q \neq l, i} \mathfrak{q}_q) \cap \mathfrak{q}'$ and $\mathfrak{q}_l \cap \mathfrak{q} \subseteq \mathfrak{q}_l$, this implies

$$(\bigcap_{q \neq l, i} \mathfrak{q}_q) \cap \mathfrak{q}' \not\subseteq \mathfrak{q}_l.$$

Hence $\mathfrak{q} = (\bigcap_{i \neq j}^n q_i) \cap \mathfrak{q}'$ is minimal primary decomposition of $\mathfrak{q}$. And $\mathfrak{q} = 0$ since

$$(\bigcap_{i \neq j}^n q_i) \cap \mathfrak{q}' = (\bigcap_{i=1}^n q_i) \cap \mathfrak{p} = 0 \cap \mathfrak{p} = 0,$$

thus $\mathfrak{p}$, which is $r(\mathfrak{q}') = r(\mathfrak{q}_i) \cap r(\mathfrak{p}) = \mathfrak{p}_i \cap \mathfrak{p} = \mathfrak{p}$, is a associate prime of 0. Hence $D(A)$ is the set of all associate primes of 0.

10. (a) If $x \in \ker(A \to A_\mathfrak{p})$, then $x/1 = 0/1$ implies $\exists s \in A - \mathfrak{p}$ such that $sx = 0$. Thus $sx \in \mathfrak{p}$, which implies $x \in \mathfrak{p}$ since $s \notin \mathfrak{p}$.

    (b) If $r(S_\mathfrak{p}(0)) = \mathfrak{p}$, then from $S_\mathfrak{p}(0) = \bigcup_{s \in S} \mathrm{Ann}(s)$,

    $$r(S_\mathfrak{p}(0)) = r(\bigcup_{s \in S} \mathrm{Ann}(s)) = \bigcup_{s \in S} r(\mathrm{Ann}(s)).$$

    Hence, $r(S_\mathfrak{p}(0)) \supseteq \mathfrak{p}$ if and only if $\forall x \in \mathfrak{p}$, $x \in \bigcup_{s \in S} r(\mathrm{Ann}(s))$, if and only if there exists $n \in \mathbb{N}, s \in S_\mathfrak{p}$ such that $sx^n = 0$ if and only if $S = \{sx^m : s \in S_\mathfrak{p}\}$ contains 0 for any $x \in \mathfrak{p} = A - S_\mathfrak{p}$, if and only if $S_\mathfrak{p}$ is a maximal multiplicative set not conatining 0, if and only if $\mathfrak{p}$ is minimal prime ideal of A by Exercise 3.6. Since $r(S_\mathfrak{p}(0)) \supseteq \mathfrak{p}$ if and only if $r(S_\mathfrak{p}(0)) = \mathfrak{p}$ by i), done.

    (c) Apply Proposition 4.12 5) in this note.

(d) 0 is definitely in $S_{\mathfrak{p}}(0)$ for all $\mathfrak{p} \in D(A)$.
Let $x \neq 0 \in S_{\mathfrak{p}}(0)$ for all $\mathfrak{p} \in D(A)$. Then, for any $\mathfrak{p} \in D(A)$, $\exists s_{\mathfrak{p}} \in A - \mathfrak{p}$ such that $xs_{\mathfrak{p}} = 0$. Thus x is zero divisor since $0 \notin A - \mathfrak{p}$ for any $\mathfrak{p}$, thus $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$ by Exercise 4.9. However, by given condition $x \in A \setminus \mathfrak{p}$, contradiction. Thus $x = 0$.

11. For the first statement, by 4.10 ii), $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$. To see $S_{\mathfrak{p}}(0)$ is primary, let $xy \in S_{\mathfrak{p}}(0)$. Then, $\exists n \in \mathbb{N}$ such that $x^n y^n \in S_{\mathfrak{p}}(0)$. Thus if $x \notin S_{\mathfrak{p}}(0)$, then $x^n \notin \mathfrak{p}$, hence $y^n \in \mathfrak{p}$, which implies $y \in \mathfrak{p} = r(S_{\mathfrak{p}}(0))$, hence $\exists m \in \mathbb{N}$ such that $y^m \in S_{\mathfrak{p}}(0)$. Since xy just arbitrary chosen, $S_{\mathfrak{p}}(0)$ is primary. To see it is the smallest $\mathfrak{p}$-primary, Apply Proposition 4.12 4).

For the second statement, let $\mathfrak{a} = \bigcap_{\mathfrak{p} \in \min(A)} S_{\mathfrak{p}}(0)$ where $\min(A)$ is a set of all minimal prime ideals of A. Notes that by Proposition 1.8, $\bigcap_{\mathfrak{p}:\text{prime}} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \min(A)} \mathfrak{p} = \mathfrak{R}$ nilradical, since every prime ideal contains at least one prime ideal in $\min(A)$. By Exercise 4.10 i),

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \in \min(A)} S_{\mathfrak{p}}(0) \subseteq \bigcap_{\mathfrak{p} \in \min(A)} \mathfrak{p} = \mathfrak{R}.$$

For the third statement, if every associate prime ideal is isolated, then $\min(A) = D(A)$, thus by Exercise 4.10 iv), $\mathfrak{a} = 0$. Conversely, if $\mathfrak{a} = 0$, then $0 = \mathfrak{a} = \bigcap_{\mathfrak{p} \in \min(A)} S_{\mathfrak{p}}(0)$ is primary decomposition of 0 by the first statement. Now just get rid of superfluous components so that for a subset Z of $\min(A)$,

$$0 = \mathfrak{a} = \bigcap_{\mathfrak{p} \in Z} S_{\mathfrak{p}}(0)$$

is the minimal primary decomposition. Thus, $Z = \text{Ass}(0) \subseteq \min(A)$.

12. (a) By Proposition 4.12 i), $S(\mathfrak{a}) = \mathfrak{a}^{ec}$. Thus,

$$S(\mathfrak{a}) \cap S(\mathfrak{b}) = \mathfrak{a}^{ec} \cap \mathfrak{b}^{ec} \underbrace{=}_{\text{Exercise 1.18}} (\mathfrak{a}^e \cap \mathfrak{b}^e)^c \underbrace{=}_{\text{Exercise 1.18}} = (\mathfrak{a} \cap \mathfrak{b})^{ec} = S(\mathfrak{a} \cap \mathfrak{b})$$

(b)

$$S(r(\mathfrak{a})) = r(\mathfrak{a})^{ec} \underbrace{=}_{\text{Prop 3.11 v)}} r(\mathfrak{a}^e)^c \underbrace{=}_{\text{Exercise 1.18}} r(\mathfrak{a}^{ec}) = r(S(\mathfrak{a})).$$

(c) If $s \in S \cap \mathfrak{a}$, then $\mathfrak{a}^e$ contains $s/1 \cdot 1/s = 1/1$, thus $\mathfrak{a}^e = S^{-1}A$, thus $S(\mathfrak{a}) = (S^{-1}A)^c = A = (1)$. Conversely, if $S(\mathfrak{a}) = (1)$, then $1/1 \in \mathfrak{a}^e$, thus $\exists a \in \mathfrak{a}$ and $s \in S$ such that $sa = 1$. This implies $s \in \mathfrak{a}$ since $s^2 a = s$ and ideal is closed under multiplication.

(d) If $x \in S_1 S_2(\mathfrak{a})$, then $\exists s_1 \in S_1, s_2 \in S_2$ such that $s_1 s_2 x \in \mathfrak{a}$. Then, $s_1 x \in S^2(\mathfrak{a})$, thus $x \in S_1(S_2(\mathfrak{a}))$. Conversely, if $x \in S_1(S_2(\mathfrak{a}))$, then $\exists s_1 \in S_1$ such that $s_1 x \in S_2(\mathfrak{a})$ which implies $\exists s_2 \in S_2$ such that $s_2 s_1 x \in \mathfrak{a}$, thus $x \in S_1 S_2(\mathfrak{a})$.

(e) For the last statement, if $\mathfrak{a} = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition, then, by Proposition 4.9, for any multiplicatively closed set S, $S(\mathfrak{a}) = \bigcap_{\mathfrak{p}_i \cap S = \emptyset} \mathfrak{q}_i$. Since all possible such intersection is finite, so the set of ideals $\{S(\mathfrak{a}) : S \text{ is multiplicatively closed set}\}$ is finite.

13. (a) Notes that $\mathfrak{p}^e$ is maximal in $A_{\mathfrak{p}}$, thus $(\mathfrak{p}^e)^n$ is $\mathfrak{p}^e$-primary. By Proposition 3.11 v), $(\mathfrak{p}^e)^n = (\mathfrak{p}^n)^e$ in this extension case. Also by [3][p.50] contraction preserves primary. thus $(\mathfrak{p}^n)^{ec}$ is primary. Now it suffices to show that its radical is $\mathfrak{p}$. To see this,

$$S(r(\mathfrak{p}^n)) = r(\mathfrak{p}^n)^{ec} \underbrace{=}_{\text{Prop 3.11 v)}} r((\mathfrak{p}^n)^e)^c \underbrace{=}_{\text{Exercise 1.18}} r((\mathfrak{p}^n)^{ec}) = r(S(\mathfrak{p}^n)).$$

Notes that $r(\mathfrak{p}^n) = \mathfrak{p}$ since $\mathfrak{p} \subseteq r(\mathfrak{p}^n)$ is clear and if $x \in r(\mathfrak{p}^n)$, then $x^m \in \mathfrak{p}^n$, thus $x \in \mathfrak{p}$ by prime property of $\mathfrak{p}$. Thus,

$$S(\mathfrak{p}) = r(S(\mathfrak{p}^n))$$

And $S(\mathfrak{p}) = \mathfrak{p}^{ec} = \mathfrak{p}$ by Corollary 3.13, a one-to-one correspondence of prime ideal of $A_{\mathfrak{p}}$ and prime ideals contained in $\mathfrak{p}$.

(b) First of all, we claim that

Claim XXIII. *$\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$.*

Proof. If $x \in \mathfrak{p}^{(n)} = S_\mathfrak{p}(\mathfrak{p}^n)$, then $\exists s \in S_\mathfrak{p}$ such that $sx \in \mathfrak{p}^n$. Thus if $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal contains $\mathfrak{p}^n$, then $sx \in \mathfrak{q}$. Since $s \notin \mathfrak{p}$, neither does s^n, thus $s, s^n \notin \mathfrak{q}$ for any $n \in \mathbb{N}$. Thus $x \in \mathfrak{q}$. Hence $\mathfrak{q} \supseteq \mathfrak{p}^{(n)}$. □

Now let $\mathfrak{p}^n = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition. Then, $\mathfrak{p} = r(\bigcap_{i=1}^m \mathfrak{q}_i) = \bigcap_{i=1}^m r(q_i)$. By Proposition 1.11 ii), $\mathfrak{p} \supseteq r(\mathfrak{q}_i)$ for some i. Since $\mathfrak{p} \subseteq r(\mathfrak{q}_i)$, this implies $\mathfrak{p} = r(\mathfrak{q}_i)$. If $r(\mathfrak{q}_j) \subsetneq \mathfrak{p}$, then

$$\mathfrak{p} = \bigcap_{i=1}^m r(q_i) \subseteq r(\mathfrak{q}_j) \subsetneq \mathfrak{p},$$

contradiction. Thus $\mathfrak{p}$ is isolated prime of $\mathfrak{p}^n$.

Now notes that $S_\mathfrak{p} \cap \mathfrak{p}_j \neq \emptyset$ for any $j \neq i$. To see this, since $\mathfrak{p}$ is isolated prime, any $\mathfrak{p}_j$ strictly contains $\mathfrak{p}$ or do not have inclusion relationship with $\mathfrak{p}$ (from the minimality of decomposition, there is no case that $\mathfrak{p}_j = \mathfrak{p}$.) This implies that $S_\mathfrak{p} = A - \mathfrak{p}$ meets $\mathfrak{p}_j$ for all $j \neq i$. Thus, by Proposition 4.9, (actually Corollary 4.11), $S_\mathfrak{p}(\mathfrak{p}^n) = \mathfrak{q}_i$. This implies $\mathfrak{p}^{(n)} = \mathfrak{q}_i$, thus it is $\mathfrak{p}$-primary component.

(c) We claim that

Claim XXIV. *$\mathfrak{p}^{(m+n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^{(n)}\mathfrak{p}^{(m)}$.*

Proof. Let $\mathfrak{q} \subseteq \mathfrak{p}^{(n)}\mathfrak{p}^{(m)}$ be a $\mathfrak{p}$-primary ideal. For any $x \in \mathfrak{p}^{(m+n)}$, there exists $s \in S_\mathfrak{p}$ such that $xs \in \mathfrak{p}^{m+n} = \mathfrak{p}^m\mathfrak{p}^n \subseteq \mathfrak{q}$. From $s \notin r(\mathfrak{q})$, $x \in \mathfrak{q}$ by primary condition. Thus $\mathfrak{p}^{(m+n)} \subseteq \mathfrak{q}$. □

By

Now suppose $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)} = \bigcap_{j=1}^m \mathfrak{q}_j$ be a minimal primary decomposition. Then, notes that by Exercise 1.13,

$$r(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) = r(\mathfrak{p}^{(m)}) \cap r(\mathfrak{p}^{(n)}) \underbrace{=}_{\text{Exercise 4.13 i)}} \mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}.$$

Hence,

$$\mathfrak{p} = r(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) = r(\bigcap_{j=1}^m \mathfrak{q}_j) = \bigcap_{j=1}^m r(\mathfrak{q}_j)$$

where last equality comes from by Exercise 1.13. By Proposition 1.11 ii), $\mathfrak{p} \supseteq r(\mathfrak{q}_i)$ for some i. Since $\mathfrak{p} \subseteq r(\mathfrak{q}_i)$, this implies $\mathfrak{p} = r(\mathfrak{q}_i)$. If $r(\mathfrak{q}_j) \subsetneq \mathfrak{p}$, then

$$\mathfrak{p} = \bigcap_{i=1}^m r(q_i) \subseteq r(\mathfrak{q}_j) \subsetneq \mathfrak{p},$$

contradiction. Thus $\mathfrak{p}$ is isolated prime of $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$.

Thus, by the same argument we did in the proof of ii), $S_\mathfrak{p}$ doesn't meet $\mathfrak{p}$ only. Thus Corollary 4.11 implies that $\mathfrak{q}_i = S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})$ Hence it suffices to show that $S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}) = \mathfrak{p}^{(m+n)}$. To see this equality, notes that if $x \in S_\mathfrak{p}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})$, then $\exists s \in S_\mathfrak{p}$ such that $sx \in \mathfrak{p}^{(m)}\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{(m+n)}$, but since $s \notin r(\mathfrak{p}^{(m+n)})$, $x \in \mathfrak{p}^{(m+n)}$ by primary condition. Conversely, if $x \in \mathfrak{p}^{(m+n)}$, then $\exists s \in S_\mathfrak{p}$ such that $sx \in \mathfrak{p}^{n+m} = \mathfrak{p}^n\mathfrak{p}^m \subseteq \mathfrak{p}^{(n)}\mathfrak{p}^{(m)}$ by claim XXI. This implies that $x/1 = a/1 \in (\mathfrak{p}^{(n)}\mathfrak{p}^{(m)})^e$, which implies $x \in S(\mathfrak{p}^{(n)}\mathfrak{p}^{(m)})$.

(d) It has a typo. What we actually do is

$$\mathfrak{p}^{(n)} = \mathfrak{p}^n \iff \mathfrak{p}^n \text{ is } \mathfrak{p}-primary.$$

This is because $\mathfrak{p}^{(n)}$ is always $\mathfrak{p}$-primary by i).

If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, then by i), $\mathfrak{p}^n$ is $\mathfrak{p}$-primary. Conversely, if $\mathfrak{p}^n$ is $\mathfrak{p}$-primary, then by the above claim XXI, $\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$, they coincides.

14. We claim that

Claim XXV. *Suppose $\mathfrak{q}$ is $\mathfrak{p}'$-primary for some prime ideal $\mathfrak{p}'$. If $(\mathfrak{q} : x)$ is a maximal ideal among all ideals of this form (for $x \notin \mathfrak{q}$), then $r(\mathfrak{q}) = (\mathfrak{q} : x)$.*

Proof. By Proposition 4.4 ii), $(\mathfrak{q} : x)$ is $\mathfrak{p}'$-primary ideal containing $\mathfrak{q}$. Let $y \in A \setminus (\mathfrak{q} : x)$. Then by definition of $(\mathfrak{q} : x)$, $xy \notin \mathfrak{q}$. Thus, By Exercise 1.12,

$$(\mathfrak{q} : x) \subseteq ((\mathfrak{q} : x) : y) = (\mathfrak{q} : xy).$$

first inclusion comes from 1.12 i), and second equality comes from 1.12 iii). By maximality with $xy \notin \mathfrak{q}$, $(\mathfrak{q} : xy) = (\mathfrak{q} : x)$. This show that for any $z \in A$, $zxy \in \mathfrak{q} \implies zx \in \mathfrak{q}$. By taking $z = y^n$, $y^{n+1}x \in \mathfrak{q}$ implies $y^n x \in \mathfrak{q}$. Thus, applying this argument $(n+1)$-times, we can conclude that $y^n x \in \mathfrak{q}$ implies $x \in \mathfrak{q}$. However, we already assume $x \notin \mathfrak{q}$, thus there is no $n \in \mathbb{N}$ such that $y^n \in (\mathfrak{q} : x)$. Hence, $y \notin r((\mathfrak{q} : x)) = \mathfrak{p}'$. To sum up, $y \in A \setminus (\mathfrak{q} : x)$ implies $y \in A \setminus \mathfrak{p}'$. In a contrapositive form, $y \in \mathfrak{p}$ implies $y \in (\mathfrak{q} : x)$. Thus

$$(\mathfrak{q} : x) \subseteq r(\mathfrak{q} : x) = \mathfrak{p}' \subseteq (\mathfrak{q} : x)$$

which implies $(\mathfrak{q} : x) = r(\mathfrak{q} : x) = \mathfrak{p}' = r(\mathfrak{q})$. $\square$

Let $\mathfrak{a} = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition with $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Suppose $x \in A \setminus \mathfrak{a}$ is such that $(\mathfrak{a} : x)$ is maximal element. By Exercise 1.12 iv)

$$(\mathfrak{a} : x) = \bigcap_{i=1}^m (\mathfrak{q}_i : x).$$

Also, by Lemma 4.4, $(\mathfrak{q}_i : x)$ is $\mathfrak{p}_i$-primary or (1). Now take $y \in (\bigcap_{j \neq i}^m \mathfrak{q}_j) \setminus \mathfrak{q}_i = (\bigcap_{j \neq i}^m \mathfrak{q}_j) \setminus \mathfrak{a}$. Then, $(\mathfrak{a} : x) \subseteq ((\mathfrak{a} : x) : y) = (\mathfrak{a} : xy)$, and maximality implies $(\mathfrak{a} : xy) = (\mathfrak{a} : x)$. Thus, we may assume that $\mathfrak{x} \in (\bigcap_{j \neq i}^m \mathfrak{q}_j) \setminus \mathfrak{a}$. Hence by Lemma 4.4,

$$(\mathfrak{a} : x) = \bigcap_{i=1}^m (\mathfrak{q}_i : x) = (\mathfrak{q}_i : x).$$

Now, if there is $y \in A \setminus \mathfrak{q}_i$ such that $(\mathfrak{q}_i : y) \supseteq (\mathfrak{q}_i : x)$, then by the same argument, we can assume that $y \in (\bigcap_{j \neq i}^m \mathfrak{q}_j) \setminus \mathfrak{q}_i$, in that case, by the Exercise 1.12 i),

$$(\mathfrak{a} : x) = (\mathfrak{q}_i : x) \subseteq (\mathfrak{q}_i : y) \supseteq (\mathfrak{q}_i : xy) = (\mathfrak{a} : xy) = (\mathfrak{a} : x)$$

implies that $(\mathfrak{q}_i : y) = (\mathfrak{q}_i : x)$. (Notes that xy is still in $(\bigcap_{j \neq i}^m \mathfrak{q}_j) \setminus \mathfrak{q}_i = (\bigcap_{j \neq i}^m \mathfrak{q}_j) \setminus \mathfrak{a}$, that's why $(\mathfrak{q}_i : xy) = (\mathfrak{a} : xy)$ by Lemma 4.4.) Thus, $(\mathfrak{q}_i : x)$ is maximal among all such forms, thus by the above claim, $(\mathfrak{q}_i : x) = r(\mathfrak{q}_i) = \mathfrak{p}_i$. Hence, $\mathfrak{p} = (\mathfrak{a} : x) = \mathfrak{p}_i$.

15. Notes that

$$S_f(\mathfrak{a}) = \{x \in A : f^n x \in \mathfrak{a} \text{ for some } n\} = \bigcup_{n>0} (\mathfrak{a} : f^n)$$

implies $S_f(\mathfrak{a}) \supseteq (\mathfrak{a} : f^n)$ for any n.

Let $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$. Assume without loss of generality, let $\Sigma = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$. Then,

$$S_f \cap \mathfrak{p}_i \neq \emptyset \iff f^n \in \mathfrak{p}_i \text{ for any } n \in \mathbb{N} \iff f \in \mathfrak{p}_i \iff \mathfrak{p}_i \notin \Sigma.$$

Thus, S_f meets only $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_n$. By Proposition 4.9,

$$S_f(\mathfrak{a}) = \bigcap_{i=1}^m \mathfrak{q}_i = \mathfrak{q}_\Sigma.$$

Thus it suffices to show that $(\mathfrak{a} : f^k)$ contains $S_f(\mathfrak{a})$ for some large k. Notes that $(\mathfrak{a} : f^n) \subseteq (\mathfrak{a} : f^{n+1})$. By Exercise 1.12 iv), we know $(\mathfrak{a} : f^k) = \bigcap_{i=1}^n (\mathfrak{q}_i : f^k)$. Thus it suffices to find n such that $\forall x \in \mathfrak{q}_\Sigma$, $f^n x \in \mathfrak{q}_i$ for all i. Now let $n_i \in \mathbb{N}$ such that $\forall x \in \mathfrak{q}_\Sigma, f^{n_i} x \in \mathfrak{q}_i$. When $\mathfrak{p}_i \in \Sigma$, such n_i exists since $x \in \mathfrak{p}_i$, so $n_i = 0$. If $\mathfrak{p}_i \notin \Sigma$, then since $f \in \mathfrak{p}_i = r(\mathfrak{q}_i)$, thus such n_i exists. Hence, by letting $n = \max_i n_i$, done.

16. For any ideal $\mathfrak{a}$, it is decomposable, thus by first three lines of the proof of Proposition 4.9 in [3], $S^{-1}\mathfrak{a}$ has a primary decomposition. Since Proposition 3.11 i) says that every ideal in $S^{-1}A$ is extended ideal of form $S^{-1}\mathfrak{a}$, done.

17. To use hint, we need to generalize Exercise 4.11.

Claim XXVI. *If $\mathfrak{p}$ is a minimal prime ideal of a ring A containing $\mathfrak{a}$, then $S_{\mathfrak{p}}(\mathfrak{a})$ is $\mathfrak{p}$-primary ideal.*

Proof. Let $f : A \to A/\mathfrak{a}$. Then, we know that $S_{\overline{\mathfrak{p}}}(\overline{0})$ is the smallest $\overline{\mathfrak{p}}$-primary ideal in $A/\mathfrak{a}$ by Exercise 4.11. Now let $\mathfrak{q} = f^{-1}(S_{\overline{\mathfrak{p}}}(\overline{0}))$. Since contraction of primary ideal is primary by [3][p.50], $\mathfrak{q}$ is primary. Also,

$$r(\mathfrak{q}) \underbrace{=}_{\text{Exercise 1.18}} f^{-1}(r(S_{\overline{\mathfrak{p}}}(\overline{0}))) = f^{-1}(\overline{\mathfrak{p}}) = \mathfrak{p}$$

shows that $\mathfrak{q}$ is $\mathfrak{p}$-primary. □

Let $\mathfrak{a}_1 = \mathfrak{a}$ for use of induction. Let $\mathfrak{p}_1$ be a minimal element of the set of prime ideals containing $\mathfrak{a}$. Then, $\mathfrak{q}_1 := S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary by above claim. By L1, $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{p}_1$. We claim that $\mathfrak{a} = q_1 \cap (\mathfrak{a} + (x))$. Notes that $\mathfrak{a} \subseteq q_1 \cap (\mathfrak{a} + (x))$ is clear. So, suppose $f \in q_1 \cap (\mathfrak{a} + (x))$. Then, $f = a + gx$ for some $a \in \mathfrak{a}, g \in A$. Also, from $\mathfrak{q}_1 = (\mathfrak{a} : x)$, $fx \in \mathfrak{a}$, which implies $ax + gx^2 \in \mathfrak{a}$. Hence, $gx^2 \in \mathfrak{a}$. From the condition $x \notin \mathfrak{p}_1 = r(\mathfrak{q}_1)$, $g \in \mathfrak{q}_1 = (\mathfrak{a} : x)$, which implies $gx \in \mathfrak{a}$, thus $f \in \mathfrak{a}$.

Let $\mathfrak{a}_2$ be a maximal element of the set of ideal $\mathfrak{b} \supseteq \mathfrak{a}_1$ such that $\mathfrak{a}_2 \cap \mathfrak{b} = \mathfrak{a}_1$ and choose $\mathfrak{a}_1$ so that $x \in \mathfrak{a}_2$, therefore $\mathfrak{a}_2 \subseteq \mathfrak{p}_1$. Such $\mathfrak{a}_2$ exists since $(\mathfrak{a} + (x)) = (\mathfrak{a}_1 + (x))$ satisfies all conditions, and if we take a collection of such ideals, then any chain (by inclusion) has a maximal element, which is union of all (by inclusion condition this union is actually an ideal) so Zorn's lemma implies that such maximal ideal exists. Say $\mathfrak{a}_2$ be such a maximal ideal. If $\mathfrak{a}_2 \neq (1)$, then do the same argument above to get $\mathfrak{q}_2, \mathfrak{p}_2$ such that $\mathfrak{a}_1 = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap (\mathfrak{a}_2 + (x_2))$ for some $x_2 \notin \mathfrak{p}_2$.

Now to use transfinite induction, suppose that $\mathfrak{a}_1 = \bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta(\mathfrak{a}_\alpha + x_\alpha)$ for some ordinal α and $x_\alpha \notin \mathfrak{p}_\alpha = r(\mathfrak{q}_\alpha)$, and $\mathfrak{a}_\alpha \not\subseteq \mathfrak{p}_\beta$ for any $\beta < \alpha$. If $\mathfrak{a}_\alpha \neq (1)$, then by the same argument above we have $\mathfrak{a}_{\alpha+1}$ containing $(\mathfrak{a}_\alpha + (x_\alpha))$ maximally and such that $\mathfrak{a}_1 = \mathfrak{a}_{\alpha+1} \cap (\bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta)$. Similarly, take $\mathfrak{p}_{\alpha+1}$ a minimal prime containing $\mathfrak{a}_{\alpha+1}$ and take $\mathfrak{q}_{\alpha+1} = S_{\mathfrak{p}_{\alpha+1}}(\mathfrak{a}_{\alpha+1})$ which is $\mathfrak{p}_{\alpha+1}$-primary, then L1 implies that $\mathfrak{q}_{\alpha+1} = (\mathfrak{a}_{\alpha+1} : x_{\alpha+1})$ for some $x_{\alpha+1} \notin \mathfrak{p}_{\alpha+1}$. By the same argument above, we have $\mathfrak{a}_{\alpha+1} = \mathfrak{q}_{\alpha+1} \cap (\mathfrak{a}_{\alpha+1} + (x_{\alpha+1}))$. Thus,

$$\mathfrak{a}_1 = \mathfrak{a}_{\alpha+1} \cap (\bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta) = \mathfrak{q}_{\alpha+1} \cap (\mathfrak{a}_{\alpha+1} + (x_{\alpha+1})) \cap (\bigcap_{\beta \leq \alpha} \mathfrak{q}_\beta) = (\mathfrak{a}_{\alpha+1} + (x_{\alpha+1})) \cap (\bigcap_{\beta \leq \alpha+1} \mathfrak{q}_\beta).$$

This is completion of successor step of the transfinite induction. For the limit step, suppose that for any $\alpha < \beta$,

$$\mathfrak{a}_1 = \mathfrak{a}_{\alpha+1} \cap (\bigcap_{\gamma < \alpha+1} \mathfrak{q}_\gamma)$$

holds and $\mathfrak{a}_\gamma \subsetneq \mathfrak{a}_\alpha$ for any $\gamma < \alpha$. Then, let $\mathfrak{a}_\beta = \bigcup_{\gamma < \beta} \mathfrak{a}_\gamma$. Then

$$\begin{aligned} \mathfrak{a}_\beta \cap \bigcap_{\rho < \beta} \mathfrak{q}_\rho &= \bigcup_{\gamma < \beta} (\mathfrak{a}_\gamma \cap \bigcap_{\rho < \beta} \mathfrak{q}_\rho) = \mathfrak{a}_1 \cup \bigcup_{\gamma+1 < \beta} (\mathfrak{a}_{\gamma+1} \cap \bigcap_{\nu < \gamma+1} \mathfrak{q}_\nu \cap \bigcap_{\gamma \leq \rho < \beta} \mathfrak{q}_\rho) \\ \underbrace{=}_{\text{From successor step}} & \mathfrak{a}_1 \cup \bigcup_{\gamma+1 < \beta} (\mathfrak{a}_1 \cap \bigcap_{\gamma \leq \rho < \beta} \mathfrak{q}_\rho) = \mathfrak{a}_1 \cup \bigcup_{\gamma+1 < \beta} (\mathfrak{a}_1) = \mathfrak{a}_1. \end{aligned}$$

Hence it holds for any ordinal.

Thus if $\mathfrak{a}_\alpha = (1)$ for some α, then done. Also, since $\mathfrak{a}_\beta$ grows at most $|A|$, so it eventually terminate at some ordinal $\beta \leq |A|$, with a decomposition $\mathfrak{a} = \mathfrak{a}_1 = \bigcap_{\gamma < \beta} \mathfrak{q}_\gamma$, which is (possibly infinitely many) primary ideals.

18. i) $\to$ L1: Let $\mathfrak{a} = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition and $\mathfrak{p}_i = r(\mathfrak{q}_i)$. Now let $\mathfrak{p}$ be a prime ideal and $S_\mathfrak{p} = A \setminus \mathfrak{p}$. Then, notes that for any $x \in A$, by Exercise 1.12, $(\mathfrak{a} : x) = \bigcap_{i=1}^m (\mathfrak{q}_i : x)$. Then let $I = \{i \in [m] : \mathfrak{q}_i \subseteq \mathfrak{p}\}$ and $J = [m] \setminus I$. Then, if $i \in I$, $S_\mathfrak{p}$ doesn't meet $\mathfrak{p}_i$. However if $i \in J$, then $S_\mathfrak{p} \cap \mathfrak{q}_i \neq \emptyset$. Thus,

$$S_\mathfrak{p}(\mathfrak{a}) = \bigcap_{i \in I} \mathfrak{q}_i.$$

Now, Lemma 4.4 shows that $(\mathfrak{q}_i : x) = (1)$ if $x \in \mathfrak{q}_i$ and $(\mathfrak{q}_i : x) = \mathfrak{q}_i$ if $x \notin \mathfrak{p}_i$. Thus if we show that there exists $x \in (\bigcap_{j \in J} \mathfrak{q}_j) \setminus \bigcup_{i \in I} \mathfrak{p}_i$, then done. To get this x, notes that for each $j \in J$, $\exists x_j \in \mathfrak{q}_j \setminus \mathfrak{p}$, thus $x = \prod_{j \in J} x_j \in (\bigcap_{j \in J} \mathfrak{q}_j \setminus \mathfrak{p}$. Since $\mathfrak{p}$ contains $\mathfrak{p}_i$ for all $i \in I$ (you can see it by taking radical) done.

i) $\to$ L2: Let $\mathfrak{a} = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition. If we let $J_n = \{i \in [m] : \mathfrak{p}_i \cap S_n = \emptyset\}$, then $S_n(\mathfrak{a}) = \bigcap_{j \in J_n} \mathfrak{q}_j$ by Proposition 4.9. By the inclusion relationship, as n increase, J_n is nonincreasing sequence of sets. Hence it converges to some set by taking $J = \bigcap_{n>0} J_n$.

ii) $\to$ i): By Exercise 4.17, we know that $\mathfrak{a} = \bigcap_{\alpha<\beta} \mathfrak{q}_\alpha$ for some ordinal $\beta \leq |A|$. Also, for each finite stage, $\mathfrak{a} = \mathfrak{a}_{n+1} \cap \mathfrak{q}_n \cap \cdots \cap \mathfrak{q}_1$, and notes that by construction $\mathfrak{a}_{n+1}$ contains $x_{n+1} \notin \mathfrak{p}_n$, thus $A - \mathfrak{p}_n = S_{\mathfrak{p}_n}$ implies $S_{\mathfrak{p}_n} \cap \mathfrak{a}_{n+1} \neq \emptyset$, thus by Exercise 4.12,

$$S_{\mathfrak{p}_n}(\mathfrak{a}) = (1) \cap S_{\mathfrak{p}_n}(\mathfrak{q}_n \cap \cdots \cap \mathfrak{q}_1) \underbrace{=}_{\text{Exercise 4.12 i)}} S_{\mathfrak{p}_n}(\mathfrak{q}_n) \cap \cdots \cap S_{\mathfrak{p}_n}(\mathfrak{q}_1) = \mathfrak{q}_n \cap \cdots \cap \mathfrak{q}_1$$

since each q_i is $\mathfrak{p}_i$-primary and by inclusion relationship of $\mathfrak{p}_i$ implies $S_{\mathfrak{p}_n}$ does not meet $\mathfrak{p}_i$, with Proposition 4.9.

Also by construction of Exercise 4.17, $\mathfrak{q}_\alpha = S_{\mathfrak{p}_\alpha}(\mathfrak{a})$. Thus, by inclusion relationship of $\mathfrak{p}_\alpha$, we know that $S_{\mathfrak{p}_1} \supseteq \cdots \supseteq S_{\mathfrak{p}_\alpha} \supseteq \cdots$ is a descending chain of multiplicatively closed subset. Now by L2, we knows that $S_n(\mathfrak{a}) = S_\alpha(\mathfrak{a})$ for any $\alpha \geq n$.

Now to see that $S_n(\mathfrak{a}) = \bigcap_{i=1}^n \mathfrak{q}_n$ is a primary decomposition, notes that

$$\mathfrak{a} = S_n(\mathfrak{a}) \cap \bigcap_{\alpha \neq 0,1,\cdots,n, \alpha<\beta} \mathfrak{q}_\alpha.$$

Now let γ be an ordinal such that $n < \gamma < \beta$. Then,

$$S_{\mathfrak{p}_\gamma}(\mathfrak{a}) \underbrace{=}_{\text{Exercise 4.12 i)}} S_{\mathfrak{p}_\gamma}(S_n(\mathfrak{a})) \cap S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\gamma) \cap \bigcap_{\alpha \neq 0,1,\cdots,n, \alpha<\beta} S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\alpha)$$

Since $S_{\mathfrak{p}_\gamma}$ meets $\mathfrak{p}_\alpha$ for $\alpha > \gamma$, so $S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\alpha) = (1)$. Thus,

$$S_{\mathfrak{p}_\gamma}(\mathfrak{a}) = S_{\mathfrak{p}_\gamma}(S_n(\mathfrak{a})) \cap S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\gamma) \cap \bigcap_{\alpha \neq 0,1,\cdots,n, \alpha<\gamma} S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\alpha)$$

Also, if $x \in S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\gamma)$, then $\exists s \in S_{\mathfrak{p}_\gamma}$ such that $sx \in \mathfrak{q}_\gamma$, which implies $x \in \mathfrak{q}_\gamma$. Conversely, $S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\gamma) = \mathfrak{q}_\gamma^{ec}$ implies $S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\gamma) \supseteq \mathfrak{q}_\gamma$, thus $S_{\mathfrak{p}_\gamma}(\mathfrak{q}_\gamma) = \mathfrak{q}_\gamma$. Also, since $\mathfrak{p}_\gamma$ contains $\mathfrak{p}_i$ for all $i \in [n]$, $S_{\mathfrak{p}_\gamma}(S_n(\mathfrak{a})) = S_n(\mathfrak{a}) = \bigcap_{i=1}^n \mathfrak{q}_i$.

Hence, with the fact that $S_{\mathfrak{p}_\gamma}(\mathfrak{a}) = S_n(\mathfrak{a})$,

$$\bigcap_{i=1}^n \mathfrak{q}_i = (\bigcap_{i=1}^n \mathfrak{q}_i) \cap \mathfrak{q}_\gamma \cap \text{some other terms}$$

thus $\mathfrak{q}_\gamma$ contains $\bigcap_{i=1}^n \mathfrak{q}_i$. Since γ was arbitrarily chosen, this implies that

$$\mathfrak{a} = \bigcap_{\alpha<\beta} \mathfrak{q}_\alpha = \bigcap_{i=1}^n \mathfrak{q}_i.$$

Hence $\mathfrak{a}$ has a primary decomposition.

19. First statement is Proposition 4.12∗ 4. in this note.

    For the second one, use proof by induction. If $n = 1$, then $\mathfrak{a} = \mathfrak{p}_1$ is such ideal, done. Suppose $n > 1$ and let $\mathfrak{p}_n$ be maximal in the set $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. By inductive hypothesis, there exists an ideal with a minimal primary decomposition $\mathfrak{b} = \bigcap_{i=1}^{n-1} \mathfrak{q}_i$ such that $r(\mathfrak{q}_i) = \mathfrak{p}_i$. We claim that $\mathfrak{b} \not\subseteq S_{\mathfrak{p}_n}(0)$. If we show this, then by Exercise 4.11 stating that $S_{\mathfrak{p}_n}(0)$ is the smallest $\mathfrak{p}_n$-primary, we have a nonempty collection of $\mathfrak{p}_n$-primary ideals not containing $\mathfrak{b}$. So taking any $\mathfrak{q}_n$ from the collection, and let $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{q}_n$. To see it is minimal prime ideal, we need a useful claim

    Claim XXVII. *Let Q be a P-primary ideal. If $IJ \subseteq Q$ and $I \not\subseteq Q$ then $J \subseteq P$.*

    Proof. Let $x \in I \setminus Q$. Then for any $y \in J$, $xy \in Q \subseteq P$. Since $x \notin Q$, $y^n \in Q$ by primary condition for some n, thus $y \in P$ since $r(Q) = P$. □

    Now, suppose $\mathfrak{q}_i \supseteq \bigcap_{j \neq i}^{n} \mathfrak{q}_j$. Let $J = \bigcap_{j \neq i}^{n-1} \mathfrak{q}_j$. Then, $J \not\subseteq \mathfrak{q}_i$ by the minimal condition of the primary decomposition of $\mathfrak{b}$. However, $J\mathfrak{q}_n \subseteq J \cap \mathfrak{q}_n \subseteq \mathfrak{q}_i$. This implies $\mathfrak{q}_n \subseteq \mathfrak{q}_i$, thus by taking radical we get $\mathfrak{p}_n \subseteq \mathfrak{p}_i$, contradicting maximality of $\mathfrak{p}_n$.

    Hence it suffices to show that $\mathfrak{b} \not\subseteq S_{\mathfrak{p}_n}(0)$. Suppose not. Let $\mathfrak{p}$ be a minimal prime ideal of A contained in $\mathfrak{p}_n$. Then $S_{\mathfrak{p}_n}(0) \subseteq S_{\mathfrak{p}}(0)$ by Proposition 4.12∗ 5 (or Exercise 10 iii)), thus $\mathfrak{b} \subseteq S_{\mathfrak{p}}(0)$. If we take radical, the by Exercise 10 ii),

    $$\bigcap_{i=1}^{n-1} \mathfrak{p}_i \subseteq \mathfrak{p}.$$

    By Proposition 1.13 ii), $\mathfrak{p}_j \subseteq \mathfrak{p}$, thus $\mathfrak{p}_j = \mathfrak{p}$ by minimality of $\mathfrak{p}$. However this contradicts our assumption that no $\mathfrak{p}_i$ is minimal prime.

20. Second equality follows from Exercise 2.2 ii) with $N + M = M$. So it suffices to show the first equality. Notes that $(N : M) = \{x \in A : xM \subseteq N\}$ by definition. Thus its radical is just $r_M(N)$, done.

    To get an analogues statements, I refer [4].

    (a) If $N \subseteq P \subseteq M$ be a submodule of M, then $r_M(N) \subseteq r_M(P)$. (Just check that $x^q M \subseteq P \subseteq N$.)
    (b) If $C \subseteq B$ are algebras over A, then $r_B(C^n) = r_B(C)$ for any $n \in \mathbb{N}$. (If $x^q B \subseteq C$, then $x^q \cdot 1 \in C$, thus $x^{qn} \cdot 1 \in C^n$, thus $x^{qn} B \subseteq C^n$.)
    (c) $r_B(\mathfrak{b}) \supseteq f^{-1}(\mathfrak{b})$ for any ideal $\mathfrak{b}$ of B and $f : A \to B$ a ring map. (If $x \in f^{-1}(\mathfrak{b})$, then $f(x) \in \mathfrak{b}$, thus $xB = f(x)B \subseteq \mathfrak{b}$.)
    (d) $r(r_M(N)) = r_M(N)$. (If $x \in r(r_M(N))$, then $x^n \in r_M(N)$ thus $x^{nq} M \subseteq N$ for some $n, q \in \mathbb{N}$, which implies $x \in r_M(N)$. The other inclusion is clear.
    (e) $r_M(N \cap P) = r_M(N) \cap r_M(P)$. ($\subseteq$ is clear. If $x \in r_M(N) \cap r_M(P)$, then $x^n M \subseteq N, x^q M \subseteq P$, thus $x^{nq} M \subseteq N \cap P$ for some $n, q \in \mathbb{N}$.)
    (f) $r_M(N) = (1) \iff M = N$ ($1M \subseteq N$ shows $\implies$ direction. Other direction is clear.)
    (g) $r_M(N + P) \supseteq r(r_M(N) + r_M(P))$. (By the first one, $r_M(N), r_M(P) \subseteq r_M(N + P)$ implies $r_M(N) + r_M(P) \subseteq r_M(N + P)$, then taking radical we get desired one.) Converse is false; let A be nonzero unital commutative ring and $M = A \oplus A$. Then, let $N = A \oplus 0, P = 0 \oplus A$, then $r_M(N + P) = (1)$ but $r_M(N) = r_M(P) = (0)$.

21. Let $xy \in (Q : M) = \mathrm{Ann}(M/Q)$. Suppose $y \notin (Q : M)$. Then, $xy(M/Q) = 0$ but $y(M/Q) \neq 0$, thus if we let $\bar{\phi}_x : M/Q \to M/Q$ by $\overline{m} \mapsto x \cdot \overline{m}$, then

    $$\bar{\phi}_x \circ \bar{\phi}_y = \bar{\phi}_{xy} = 0.$$

    Since $\bar{\phi}_y$ is nonzero, $\ker \bar{\phi}_x \supseteq \mathrm{Im}(\bar{\phi}_y) \neq 0$. Hence $\bar{\phi}_x$ has nonzero kernel, thus x is zero divisor of Q. Since Q is primary, x is nilpotent, thus $\exists n \in \mathbb{N}$ such that $\bar{\phi}_{x^n} = 0$. Thus, $x^n M \subseteq Q$, which implies $x^n \in (Q : M)$. Thus $(Q : M)$ is primary.

Lemma 4.3∗. *If Q_i is $\mathfrak{p}$-primary for $i = 1, 2, \cdots, n$ submodules of M, then So is $Q = \bigcap_{i=1}^n Q_i$.*

Proof. Let $\overline{x} \in M/Q$ be a zero divisor. Then, $\exists \overline{y} \neq 0$ such that $\overline{xy} = 0$. This implies $xy \in Q = \bigcap_{i=1}^n Q_i$. Since $\overline{x}$ is zero divisor on each M/Q_i, it is nilpotent, thus $x^{n_i} M \subseteq Q_i$ for some n_i for each i. Thus, let $n = \sum_i n_i$, then $x^n M \subseteq Q$, which implies that x is nilpotent on M/Q. To see it is $\mathfrak{p}$-primary,

$$r(Q : M) = r(\bigcap Q_i : M) = r(\bigcap (Q_i : M)) = \bigcap r(Q_i : M) = \bigcap \mathfrak{p} = \mathfrak{p}$$

where $\bigcap Q_i : M = \bigcap (Q_i : M)$ is application of generalization of 1.12 iv). To see this, if $x \in \bigcap Q_i : M$, then $xM \subseteq \bigcap Q_i$, thus $x \in (Q_i : M)$ for all i. The other direction is similar. □

Proposition 4.4∗. *Let $Q \subseteq M$ be a $\mathfrak{p}$-primary submodule of M. Then,*

(a) if $x \in Q$ then $(Q : x) = (1)$;

(b) if $x \notin Q$ then $(Q : x)$ is $\mathfrak{p}$-primary, and therefore $r(Q : x) = \mathfrak{p}$;

(c) if $x \notin \mathfrak{p}$ then $(Q : x) := \{m \in M : xm \in Q\} = Q$.

Proof. i) is clear. For ii), let $yz \in (Q : x)$ but $y \notin (Q : x)$. Then $yz.x \in Q$ but $y.x \notin Q$, so $\overline{z}$ is zero divisor of M/Q, hence nilpotent. This implies $\bar{\phi}_{z^n} = 0$ as an endomorphism on M/Q. Thus $z^n M \subseteq Q$, thus $z^n \in (Q : x)$. This implies $(Q : x)$ is primary. To see it is $\mathfrak{p}$-primary, notes that $(Q : M) \subseteq (Q : x)$ implies $\mathfrak{p} \subseteq r(Q : x)$. Also, if $a \in r(Q : x)$, then $a^n \in (Q : x)$ for some $n \in \mathbb{N}$, thus $a^n x \in Q$, which implies a is zero divisor on M/Q, thus by primary condition a is nilpotent, i.e., $\bar{\phi}_{a^k} = 0$ as an endomorphism on M/Q. This shows that $a^k \in r_M(Q) = \mathfrak{p}$, thus $a \in r_M(Q) = \mathfrak{p}$ since it is radical.

For iii), if $m \notin Q$ but $xm \in Q$, then x is zero divisor on M/Q, thus nilpotent, hence $\bar{\phi}_{x^n} = 0$ as an endomorphism on M/Q. Thus, $x \in r_M(Q) = \mathfrak{p}$, contradiction. Hence $(Q : x) \subseteq Q$. The other way of inclusion is clear. □

22.

Theorem 4.5∗. *If $N = Q_1 \cap \cdots \cap Q_n$ is a minimal primary decomposition with $\mathfrak{p}_i = r_M(Q_i)$,then $\mathfrak{p}_i$ are precisely the prime ideals which occur in the set of ideals $\{r(N : m) | m \in M\}$ hence are independent of the particular decomposition of N.*

Proof. Fix $i \in [n]$ and let $m \in \bigcap_{j \neq i} Q_j \setminus Q_i$. Such m exists since the given decomposition is minimal. Then,

$$(N : m) = \bigcap_j (Q_j : m) = (Q_i : m)$$

where equality comes from the above argument similar to 1.12 iv). By Proposition 4.4∗, $(Q_j : m) = (1)$ when $j \neq i$. Thus $(N : m)$ is $\mathfrak{p}_i$-primary, so each $\mathfrak{p}_i = r(N : m)$ for some $m \in M$.

Conversely, let $r(N : m)$ is prime $\mathfrak{p}$ for some $m \in M$. Then, $(N : m) = \bigcap (Q_i : m)$ shows that

$$r(N : m) = \bigcap_{m \notin Q_i} \mathfrak{p}_i.$$

By Exercise 1.11 i) and ii), with the assumption that $r(N : m)$ is prime, $r(N : m) = \mathfrak{p}_i$ for some i. □

23. Last comments in [3] implies that for any decomposable submodule N of M, with a minimal primary decomposition $N = \bigcap_{i=1}^n Q_i$, Q_i/N is $r_{M/N}(Q_i/N)$-primary module with $r_{M/N}(Q_i/N) = r_M(Q_i)$. To see this, first of all, $(Q_i/N : M/N) = \{x \in A : xM/N \subseteq Q_i/N\} = \{x \in A : xM \subseteq Q_i\} = (Q_i : M)$ gives $r_{M/N}(Q_i/N) = r_M(Q_i)$. And to see Q_i/N is primary, let x be a zero divisor on $(M/N)/(Q_i/N) \cong M/Q_i$. Then, x is zero divisor of M/Q_i, thus nilpotent on M/Q_i, therefore nilpotent on $(M/N)/(Q_i/N)$, done. By the same argument in different direction, if Q_i/N is $r_{M/N}(Q_i/N)$-primary, then Q_i is $r_M(Q_i)$-primary.

Proposition 4.6∗. *Let N be a decomposable submodule of M. Then any prime ideal $\mathfrak{p} \supseteq r_M(N)$ contains a minimal prime ideal belonging to N, and thus the minimal prime ideals of N are precisely the minimal elements in the set of all prime ideals containing N*

Proof. If $N = \bigcap_{i=1}^n Q_i$, then $r_M(N) = \bigcap_{i=1}^n r_M(Q_i) = \bigcap_{i=1}^n \mathfrak{p}_i$ where Q_i is $\mathfrak{p}_i$-primary module. By Proposition 1.11 ii), from $\mathfrak{p} \supseteq r_M(N) = \bigcap_{i=1}^n \mathfrak{p}_i$, $\mathfrak{p}$ contains $\mathfrak{p}_j$ for some $j \in [n]$. Hence $\mathfrak{p}$ contains some minimal prime ideal of N. ☐

Proposition 4.7∗. *Let N be a decomposable submodule of M, let $N = \bigcap_{i=1}^n Q_i$ be a minimal primary decomposition, and let $r_M(Q_i) = \mathfrak{p}_i$. Then*

$$\bigcup_{i=1}^n \mathfrak{p}_i = \{x \in A : (N : x) \neq N\}.$$

In particular, if the zero module is decomposable, the set D of zero-divisors of M is the union of the prime ideal belonging to 0.

Proof. By the above generalization argument, and the fact that for any $x \in A$, $(0 : x) \neq 0$ in M/N implies $(N : x) \neq N$ in M wh we can assume $N = 0$ (otherwise we can think everything on M/N.) Then let $D' = \{x \in A : (0 : x) \neq 0\}$ the right hand side. If $(0 : x) \neq 0$ for some x, then there exists $m \in (0 : x)$, with $m \neq 0$ but $xm = 0$. Thus x is zero divisor in a sense of Exercise 4.21. Thus, $D \supseteq D'$. Conversely, if x is zero divisor, then $\exists m \neq 0 \in M$ such that $xm = 0$, thus $m \in (0 : x)$ implies $(0 : x) \neq 0$, thus $x \in D'$. This shows that $D' = D$.

Also, if $x \in r(D)$, then $x^n m = 0$ for some nonzero m and some $n \in \mathbb{N}$, thus pick n the smallest such that $x^n m = 0$, we can see that $x^{n-1}m \neq 0$, which implies $x \in D$. Hence $r(D) = D$. Thus,

$$D = r(D) = r(\bigcup_{m \neq 0} (0 : m)) = \bigcup_{m \neq 0} r(0 : m).$$

(Since it is union of set, so we can apply argument on chapter 1.) By the first uniqueness theorem (Exercise 4.22), all prime ideal belonging to 0 has a form $r(0 : m)$. Thus D contains union of prime ideals belonging to 0. Also, by the proof of Exercise 4.22, each $r(0 : m)$ is intersection of some prime ideals belonging to 0, thus contained in a prime ideal belonging to 0, which shows that D is equal to union ot prime ideals belonging to 0. ☐

Proposition 4.8∗. *Let S be a multiplicatively closed subset of A, and let Q be a $\mathfrak{p}$-primary module of M.*

(a) If $S \cap \mathfrak{p} \neq \emptyset$, then $S^{-1}Q = S^{-1}M$.
(b) If $S \cap \mathfrak{p} = \emptyset$, then $S^{-1}Q$ is $S^{-1}\mathfrak{p}$-primary and its contraction in M is Q.

Hence primary module of $S^{-1}M$ corresponds to primary modules of M.

Proof. For the first one, if $s \in S \cap \mathfrak{p}$, then $s^n M \subseteq Q$ for some $n \in \mathbb{N}$. Hence, for any $m/t \in S^{-1}M$, $m/t = s^n m/s^n t \in S^{-1}Q$, hence $S^{-1}M = S^{-1}Q$.

For the second one, let $\overline{x/s} \in \overline{S^{-1}A}$ is zero divisor of $S^{-1}M/S^{-1}Q$. Then, there exists nonzero $\overline{m/t} \in S^{-1}M/S^{-1}Q$ such that $\overline{xm/st} = 0$. Thus $xm/st \in S^{-1}Q$, there exists $u \in S$ such that $uxm \in Q$. Since $m \notin Q$, ux is zero divisor of M/Q, thus ux is nilpotent of M/Q since Q is primary. Hence, $\overline{m} \mapsto (ux)^k \overline{m} = 0$ for any $\overline{m} \in M/Q$. Hence,

$$x^k S^{-1}M = x^k u^k S^{-1}M \subseteq S^{-1}Q.$$

This implies x is nilpotent in $S^{-1}M/S^{-1}Q$. Thus $S^{-1}Q$ is primary module.

Let $x \in \mathfrak{p} = r_M(Q)$. Then, $x^k M \subseteq Q$ for some k. Thus, for any $s \in S$, $(x/s)^k S^{-1}M = x^k S^{-1}M = S^{-1}(x^k M) \subseteq S^{-1}Q$. Hence, $x/s \in r_{S^{-1}M} S^{-1}Q$ for any $s \in S$. This implies that $S^{-1}\mathfrak{p} \subseteq r_{S^{-1}M} S^{-1}Q$.

Conversely, if $x/s \in r_{S^{-1}M}S^{-1}Q$, then $\exists k \in \mathbb{N}$ such that $(x/s)^k S^{-1}M \subseteq S^{-1}Q$, which implies that $(x/s)^k S^{-1}M = x^k S^{-1}M \subseteq S^{-1}Q$. Thus for any $m \in M$, $x^k m/1 \in S^{-1}Q$ implies $\exists t \in S$ such that $tx^k m \in Q$. Then tx^k is zero divisor of M/Q, thus nilpotent, i.e., $t^n x^{kn} m \in Q$ for any $m \in M$, thus $t^n x^{kn} \in r_M(Q)$, which implies $tx^k \in r_M(Q) = \mathfrak{p}$ by radical property. Since $t \notin \mathfrak{q}$ by construction, $x^k \in \mathfrak{p}$, which implies $x \in \mathfrak{p}$. Thus, for any $x/s \in r_{S^{-1}M}S^{-1}Q$, $x \in \mathfrak{p}$. This implies that $r_{S^{-1}M}S^{-1}Q \subseteq S^{-1}\mathfrak{p}$, thus they are equal. This shows that $S^{-1}Q$ is $S^{-1}\mathfrak{p}$-primary.

Lastly, it suffices to show that every submodule of $S^{-1}M$ is extended module, i.e., $S^{-1}N$ for some submodule N of M. Let N' be a submodule of $S^{-1}M$. Then, let $N = \{m \in M : m/1 \in N'\}$, i.e., contraction of N' along $M \to S^{-1}M$. Then, $S^{-1}N \supseteq N'$, since for any $m/s \in N'$, $sm/s = m/1 \in N'$, thus $m \in N$, which implies $m/s \in S^{-1}N$. Conversely, if $x/s \in S^{-1}N$, then $x \in N$, thus $x/1 \in S^{-1}N$, therefore $(1/s) \cdot (x/1) = x/s \in N'$. This shows that $N' = S^{-1}N$.

Thus, extension of primary ideal is primary. Conversely, let $S^{-1}Q$, a submodule of $S^{-1}M$, be $S^{-1}\mathfrak{p}$-primary ideal. (We can say this since every prime ideal in $S^{-1}A$ is extended form.) Then, it suffices to show that Q is $\mathfrak{p}$-primary ideal. From the map $f : M \to N$ and Q is primary ideal of N, then M/Q^c is isomorphic to the submodule of N/Q by map f, thus every zero divisor of M/Q^c is also a zero divisor of N/Q, thus nilpotent in N/Q, which implies nilpotent in M/Q^c as a submodule of N/Q. To see it is $\mathfrak{p}$-primary, the above argument exactly applicable in this situation, since it use only property of $r_M(Q)$ and $r_{S^{-1}M}(S^{-1}Q)$. □

Proposition 4.9∗. *Let S be a multiplicatively closed subset of A and let $N \subseteq M$ be a decomposable module. Let $N = \bigcap_{i=1}^m Q_i$ be a minimal primary decomposition of N. Let $\mathfrak{p}_i = r_M(Q_i)$ and suppose the Q_i numbered so that S meets $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_n$ but not $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$. Then,*

$$S^{-1}N = \bigcap_{i=1}^m S^{-1}Q_i, \quad S(N) = \bigcap_{i=1}^m Q_i.$$

and these are minimal primary decompositions.

Proof. First equality comes from Corollary 3.4 ii) with Proposition 4.8∗ above. ($S^{-1}Q_i = S^{-1}M$ for any $i \in \{m+1, \cdots, n\}$.) Since $S^{-1}Q_i$ is $S^{-1}\mathfrak{p}_i$-primary by Proposition 4.8∗, it is primary decomposition. Also, each $S^{-1}\mathfrak{p}_i$ are distinct, and if $\exists j \in [n]$ such that $S^{-1}Q_j \supseteq \bigcap_{i\neq j}^m S^{-1}Q_i$, then

$$Q_j = f^{-1}(S^{-1}Q_j) \supseteq f^{-1}\left(\bigcap_{i\neq j}^m S^{-1}Q_i\right) = \bigcap_{i=1}^m Q_i$$

contradiction. Thus it is minimal primary decomposition.

For $S(N) = f^{-1}\left(\bigcap_{i\neq j}^m S^{-1}Q_i\right) = \bigcap_{i=1}^m Q_i$, first of all it is primary decomposition with distinct prime. Also, if $Q_j \supseteq \bigcap_{i\neq j}^m Q_i$, then Q_j contains $\bigcap_{i\neq j}^n Q_i$, contradicting the assumption that $\bigcap_{i=1}^n Q_i$ is the minimal primary decomposition. Thus, $\bigcap_{i=1}^m Q_i$ is minimal primary decomposition of $S(N)$. □

Theorem 4.10∗. *Let $N \subseteq M$ be a decomposable module, let $N = \bigcap_{i=1}^n Q_i$ be a minimal primary decomposition of N, and let $\{\mathfrak{p}_{i_1}, \cdots, \mathfrak{p}_{i_m}\}$ be an isolated prime ideal of N. Then $\bigcap_{j=1}^m Q_{i_j}$ is independent of the decomposition.*

Proof. Let $S = A \setminus \bigcup_{j=1}^m \mathfrak{p}_{i_j}$. Then, S doesn't meet $\mathfrak{p}_{i_j}$ for any $j = 1, \cdots, m$. For any other prime ideal $\mathfrak{p}_k$, $\mathfrak{p}_k \notin \{\mathfrak{p}_{i_1}, \cdots, \mathfrak{p}_{i_m}\}$ implies $\mathfrak{p}_k \not\subseteq \bigcup_{j=1}^m \mathfrak{p}_{i_j}$ by Proposition 1.11 i), thus S meets $\mathfrak{p}_k$. This implies $S(N) = \bigcap_{j=1}^m Q_{i_j}$ by Proposition 4.9∗ and it is independent to the choice of minimal primary decomposition. □

Corollary 4.11∗. *The isolate primary components (i.e., the primary components Q_i corresponding to minimal prime ideal $\mathfrak{p}_i$) are uniquely deteremined by N.*

Proof. No associate prime contains the minimal prime ideal except itself, thus by letting $S = A \setminus \bigcup_{j\neq i} \mathfrak{p}_j$, we can get $S(N) = Q_i$ by Proposition 4.9∗. It is independent of choice of minimal primary decomposition. □

5 Integral Dependence and Valuations

We mention some missing steps in the book.

First of all, in [3][p.60], it mentions finite type + integral = finite. We can reveal the details behind it.

Claim XXVIII. *Let $f : A \to B$ be a ring homomorphism. Then f is of finite if and only if f is finite type and integral.*

Proof. Suppose f is finite type and integral. Then, there exists $b_1, \cdots, b_n \in B$ such that every elements in B is polynomial of b_is with coefficients in A. Thus, $B = A[b_1, \cdots, b_n]$. Corollary 5.2 says that B is a finitely generated A-module, thus f is of finite.

Conversely, suppose f is of finite. Then, B is a finitely generated A-module, hence B has a generators $b_1, \cdots, b_n$ such that every elements of B can be written as linear combination of those. Since linear combination is also a polynomial, so B is of finite type. To see B is integral over A, notes that $A[b_i]$ is contained in a ring B which is finitely generated A-module for each i. Thus, b_i is integral over $f(A)$, for all i. Hence, $B = A[b_1, \cdots, b_n]$ contained in a set of elements of B which are integral over B by Corollary 5.3. This implies that B is integral over A. $\square$

Proposition 5.15. *See book.*

In the proof, "The coefficients of the minimal polyonmial of x over K are polynomials in the x_i" comes from Viete's theorem.

Theorem 5.16. *Going down theorem.*

In the proof, it assumes that $x = B_{\mathfrak{q}_1}\mathfrak{p}_2 \cap A$. Then, $s = yx^{-1}$ comes from the fact that 1) s as an elements of the field of fraction of B, say K'. Since $A \subseteq B$, $K' \supseteq K$, thus $x, y, x^{-1} \in K'$, therefore $s = yx^{-1}$, since $x = y/s$ implies $s = y/x$ in K'.

Also the book concludes that $B_{\mathfrak{q}_1}\mathfrak{p}_2 \cap A \subseteq \mathfrak{p}_2$ implies they are equal. The other direction comes from the fact that $B_{\mathfrak{q}_1}\mathfrak{p}_2 = \mathfrak{p}_2^{ec}$ and $\mathfrak{a}^{ec} \supseteq \mathfrak{a}$ by Proposition 1.17.

Lemma 5.19. *B is a local ring and $\mathfrak{m} = \ker(g)$ is its maximal ideal.*

In the proof, we can identify B as an elements of $B_{\mathfrak{m}}$. Thus, $B = B_{\mathfrak{m}}$ implies that every elements $1/s \in B$ for any $s \in A - \mathfrak{m}$, which implies that every elements outside of $\mathfrak{m}$ is unit. Thus, by Proposition 1.6 B is a local ring.

Lemma 5.20. *See the textbook.*

If $\mathfrak{m}[x] = B[x]$, then $1 \in \mathfrak{m}[x]$, thus $1 = \sum_{j=1}^{k} m_j f_j$ for some $f_j \in B[x]$, and any f_j can be denoted as polynomial of x over B, thus 1 is sum of polynomials over x whose coefficients are all from $\mathfrak{m}$.

Theorem 5.21. *Let (B, g) be a maximal element of Σ. Then B is a valuation ring of the field K.*

In the proof, notes that unit in $1_B \notin \mathfrak{m}'$ thus $\mathfrak{m}' \cap B$ is a proper ideal of B containing $\mathfrak{m}$, thus by maximality, $\mathfrak{m} = \mathfrak{m}' \cap B$.

Also, the author claimed that $k' = k[\overline{x}]$ which is a subring generated by k and $\overline{x}$ implies $\overline{x}$. To see this, we need a claim.

Claim XXIX. *Let A be an integral domain that is finitely generated over a field K. If A is a field, then A is algebraic over K.*

Since $k[\overline{x}]$ is a field, by the claim $\overline{x}$ is algebraic over k.

Proof. To use induction, let $A = k[z_1]$. If z_1 is not algebraic, then it is transcendental, thus $k[x] \cong k[z_1]$ but $k[x]$ is not a field, contradiction. Now suppose that $A = k[z_1, \cdots, z_n]$. Then, since A is a field, $A = k[z_1][z_2, \cdots, z_n] = k(z_1)[z_2, \cdots, z_n]$. Hence, by inductive hypothesis, $z_2, \cdots, z_n$ are algebraic over $k(z_1)$. Thus, for any $2 \leq i \leq n$, there exists $f_j \in k(z_1)[x]$ such that $f_j(z_j) = 0$. Now we can rewrite

$$f_j = B_j x^{n_j} + \text{ lower order terms}$$

such that $B_j \in k(z_1)$. By multiplying their product of all denominators of coefficients, we can assume that every coefficients of f_j is in $k[z_1]$. Let $B = \prod_{j=2}^{n} B_j$ and define $S = k[z_1, B]$. Notes that $B_j \in S$ for all $2 \le j \le n$. Also,

$$A \supseteq S[z_2, \cdots, z_n] \supseteq k[z_1, \cdots, z_n] = A$$

implies $S[z_2, \cdots, z_n] = A$. Thus, $f_j/A_j = x^{n_j} +$ lower terms $\in S[x]$, and $f_j(z_j)/A_j = 0$. Hence, z_j is algebraic over S, which implies that z_j is algebraic over A since $A = S$. $\square$

And in the proof of Corollary 5.22, the author needs a claim that

Claim XXX. *If A is a subring of a field K with homomorphism $f : A \to \Omega$, then there exists a valuation ring B such that $A \subseteq B \subseteq K$ and homomoprhism $g : B \to \Omega$ such that $g|_A = f$.*

Proof. Let $\Sigma' = \{(B', f) \in \Sigma : (B', f) \supseteq (A, f)\}$. Then, it is nonempty since (A, f) is in Σ'. Also, any chain in this collection has a maximal one by unioning all elements in the chain. (By given condition they are glued well.) Thus, by the Zorn's lemma it has a maximal elements $(B.g)$. Notes that this maximal elements is also a maximal elemtns of Σ, otherwise $\exists (B', g')$ such that $(A, f) \le (B.g) \le (B', g')$, thus $(B', g') \in \Sigma'$, therefore $(B', g') = (B, g)$. Hence by applying theorem 5.21, we know that (B, g) is a valuation ring containing (A, f). Also, this map g is extension of f. $\square$

Thus, using (5.21) in the proof of Proposition 5.23 is actually using the above claim.

Definition . *Integral homomorphism $f : A \to B$ is a ring homomorphism such that B is integral over $f(A)$.*

1. We can factor through f as
$$A \xrightarrow{r} f(A) \xrightarrow{s} B.$$
Then, r is surjection and s is injection. By Exercise 1.21 iv), r^* is homeomorphism of $\mathrm{Spec}(f(A))$ onto $V(\ker(r))$. Thus, it is closed map. So it suffices to show that s^* is closed map.

   Let $\mathfrak{b}$ in B. Then, $\mathfrak{b} \cap f(A)$ is a contraction of $\mathfrak{b}$ with respect to i. It suffices to show that $s^*(V(\mathfrak{b})) = V(\mathfrak{b} \cap f(A))$. Let $\mathfrak{p}$ be a prime ideal of B containing $\mathfrak{b}$. Then, $s^*(\mathfrak{p}) = \mathfrak{p} \cap f(C)$ contains $\mathfrak{b} \cap f(A)$, thus $s^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b} \cap f(A))$.

   Conversely, if $\mathfrak{p}$ is a prime ideal in $\mathrm{Spec}(f(A))$ containing $\mathfrak{b} \cap f(A)$, $\overline{\mathfrak{p}}$ is prime ideal over $f(A)/\mathfrak{b} \cap f(A)$. Proposition 5.6 says that $B/\mathfrak{b}$ is integral over $f(A)/\mathfrak{b} \cap f(A)$. Thus by Theorem 5.10 $\exists \overline{\mathfrak{q}}$ a prime ideal such that $\overline{q} \cap f(A)/\mathfrak{b} \cap f(A) = \overline{\mathfrak{p}}$. If we let $\phi : f(A)/\mathfrak{b} \cap f(A) \to B/\mathfrak{b}$ is a canonical injective homomorphism, then theorem 5.10 implies that ϕ^* is surjective continuous map. Since $\mathrm{Spec}(B/\mathfrak{b}) \cong V(\mathfrak{b}) \subseteq \mathrm{Spec}(B)$ and $\mathrm{Spec}(f(A)/\mathfrak{b} \cap f(A)) \cong V(\mathfrak{b} \cap f(A)) \subseteq \mathrm{Spec}(f(A))$,
$$V(\mathfrak{b}) \cong \mathrm{Spec}(B/\mathfrak{b}) \xrightarrow{\phi^*} spec(f(A)/\mathfrak{b} \cap f(A)) \cong V(\mathfrak{b} \cap f(A))$$
is surjective continuous map. By Exeercise 3.21 iii), this long map is a restriction of s^* on $V(\mathfrak{b})$. Thus, $s^*(V(\mathfrak{b})) = V(\mathfrak{b} \cap f(A))$.

2. Notes that $f(A)$ is a subring of Ω. Thus, $f(A) \cong A/\ker(f)$, and $\ker(f)$ is a maximal ideal of A. Say $\mathfrak{m} = \ker(f)$. Then, by Theorem 5.10 there exists a prime ideal $\mathfrak{m}'$ of B such that $\mathfrak{m}' \cap A = \mathfrak{m}$. Hence, $B/\mathfrak{m}'$ is integral over $f(A)$. Thus, if we find out an injective homomorphism $\overline{g} : B/\mathfrak{m}' \to \Omega$ which is extension of $\overline{f} : A/\mathfrak{m} \to \Omega$, then $g : B \to B/\mathfrak{m}' \to \Omega$ is an extension of f since for any $a \in A \subseteq B$, $g(a) = \overline{g}(a + \mathfrak{m}') = \overline{f}(a + \mathfrak{m}) = f(a)$.

   Thus just assume that A, B are integral domain. Let $\Sigma = \{(C, h) : A \subseteq C \subseteq B, h|_A = f\}$ be a set of all pair of integral domains C and an extension of f over C. Define order $(C, h) \le (C', h')$ if $C \subseteq C'$ an $h'|_C = h$. Then, $(A, f) \in \Sigma$, and for any chain, the union of all rings in the chain and maps give well-defined integral domain and maps into Ω. Thus, by Zorn's lemma, there exists a maximal element (C, h). If $C \neq B$, then $\exists b \in B \setminus C$. Since B is integral over A, b is integral over A, thus integral over C. Hence, $0 = \sum_{i=1}^{n} c_i b^i$ for some $c_i \in C$. Suppose that n is minimal among all possible polynomials having c as its zero. Then, $\phi(x) = \sum_{i=1}^{n} h(c_i) x^i \in \Omega[x]$. Also $\Omega[x]/(\phi(x)) \cong \Omega(c)$ is a field from the basic fact about Field extension. Since Ω is algebraically closed, $\Omega(c) = \Omega$. Hence,
$$\rho : C[x] \xrightarrow{h} \Omega[x] \to \Omega[x]/(\phi(x)) \cong \Omega$$

is a homomorphism with a kernel $(\sum_{i=1}^n c_i x^i)$, and it is extension of h. Thus induce an injection

$$C[x]/(\sum_{i=1}^n c_i x^i) \to \Omega.$$

Since $C[x]/(\sum_{i=1}^n c_i x^i) \cong C[b]$ as a ring, (since b satisfies $\sum_{i=1}^n c_i x^i$) this induces an injection from $C[b]$ to Ω extending h. Since $C[b]$ is strictly bigger than C, it contradicts maximality of (C, h). Hence $C = B$.

3. Let $b \otimes c \in B' \bigotimes_A C$. Then, since B' is integral over $f(B)$, $\sum_{j=1}^n b_j b^j = 0$ for some $b_j \in f(B)$. Hence, $(\sum_{j=1}^n b_j b^j) \otimes c = 0$, which implies that $(\sum_{j=1}^n b_j b^j) \otimes c^n = 0$ by acting c^{n-1} on 0. From this we can get

$$(\sum_{j=1}^n b_j \otimes c^{n-j}) \cdot (b \otimes c)^j = 0.$$

Thus, $b \otimes c$ is integral over $f \otimes 1(B \bigotimes_A C)$ since each coefficient is in the image. By definition of tensor product, every elements in the tensor product is finite sum of basic elements of a form $b \otimes c$, thus $B' \bigotimes_A C$ is integral over $B \bigotimes_A C$ with respect to $f \otimes 1$.

4. No. Suppose $\operatorname{char} k \neq 2$, let $B = k[x], A = k[x^2 - 1]$, then B is integral over A since $f(y) = y^2 - 1 - (x^2 - 1)$ is a polynomial over A having x as its zero. Let $\mathfrak{n} = (x - 1)$ be a maximal ideal of B. Now $1/(x + 1)$ is an element in $B_\mathfrak{n}$, since $x + 1 \notin \mathfrak{n}$. If it is integral over $A_\mathfrak{m}$, where $\mathfrak{m} = \mathfrak{n} \cap A = (x^2 - 1)$, then

$$\sum_{j=1}^n (\frac{a_j}{s_j})(x + 1)^{-j} = 0$$

for some $a_j / s_j \in A_\mathfrak{m}$ with $a_n = 1$. By multiplying $(x + 1)$ on each side of fraction we can assume that

$$\frac{\sum_{j=1}^n a_j (x + 1)^{n-j}}{s_j (x + 1)^n} = 0$$

for different a_j's. Since B is integral domain, $a_n t_n + \sum_{j=1}^{n-1} a_j t_j (x + 1)^{n-j} = \sum_{j=1}^n a_j t_j (x + 1)^{n-j} = 0$ in B where $t_j = \prod_{i \neq j} s_j$ by multiplying $(x + 1)^n \prod_{j=1}^n s_j$. Since $s_j \in A \setminus \mathfrak{m}$, $t_j \in A \setminus (x^2 - 1)$, thus $(x + 1)$ divides $\sum_{j=1}^{n-1} a_j t_j (x + 1)^{n-j}$. This implies that $a_n t_n \in (x + 1) \subseteq B$. And since we assume that $a_n = 1$, this implies $t_n \in (x + 1)$. Also, since $t_n \in A$, $t_n \in (x + 1) \cap A = (x^2 - 1)$, contradiction since $A - (x^2 - 1)$ is multiplicatively closed.

5. (a) Since $x^{-1} \in B$, then $0 = \sum_{j=1}^n a_j x^{-j}$ with $a_j \in A$, $a_n = 1$. Hence by multiplying x^{n-1}, we get

$$x^{-1} + \sum_{j=1}^{n-1} a_j x^{n-1-j} = 0$$

thus x^{-1} is linear combination of x^i over A. Hence, it is in A.

(b) Let $\max(A)$ be a collection of all maximal ideals of A. Then, for any $\mathfrak{m} \in \max(A)$, Theorem 5.10 implies that $\exists \mathfrak{n} \in \operatorname{Spec}(B)$ such that $\mathfrak{n} \cap A = \mathfrak{m}$. By Corollary 5.8, $\mathfrak{n}$ is also a maximal ideal. Conversely, if $\mathfrak{n} \in \max(B)$ is a maximal ideal of B, then $\mathfrak{n} \cap A \neq A$ since 1 is not in $\mathfrak{n}$, thus it is proper subideal of A (since $\mathfrak{n} \cap A = \mathfrak{n}^c$) therefore it is maximal by Corollary 5.8. Thus, we can conclude that

Claim XXXI. *If $A \subseteq B$ be rings and B is integral over A, then* $\max(B)$ *and* $\max(A)$ *has 1-1 correspondence given by extension and contraction.*

Hence, if we denote $\mathfrak{R}_A, \mathfrak{R}_B$ be the Jacobson radical of A and B respectively, then,

$$\mathfrak{R}_A = \bigcap_{\mathfrak{m}\in\max(A)} \mathfrak{m} = \bigcap_{\mathfrak{n}\in\max(B)} \mathfrak{n} \cap A = A \cap \left(\bigcap_{\mathfrak{n}\in\max(B)} \mathfrak{n} \right) = A \cap \mathfrak{R}_B.$$

6. Let $(b_1, \cdots, b_n) \in \prod_{i=1}^n B_i$. Then, for each b_i, there exists a monic polynomial

$$f_i(x) = x^{n_i} + \sum_{j=1}^{n_i-1} a_{ij} x^j = 0.$$

having b_i as zero. Thus, let $f(x) = \prod_{i=1}^n f_i(x)$. Then, $f(x)$ is monic and has $b_1, \cdots, b_n$ as its solutions. Now we can rewrite $f = \sum_{j=1}^m a_j x^j$ with $a_m = 1$. Now, notes that multiplication in $\prod_{i=1}^n B_i$ is coordinatewise multiplication (thinking a_j as scalar.) Thus, we can replace x of $f(x)$ with $(b_1, \cdots, b_n)$. Then,

$$f(b_1, \cdots, b_n) = (f(b_1), \cdots, f(b_n)) = (0, \cdots, 0)$$

and $(0, \cdots, 0)$ is zero in $\prod_{i=1}^n B_i$. Thus, $(b_1, \cdots, b_n)$ is integral over A, and was arbitrarily chosen, thus $\prod_{i=1}^n B_i$ is an integral A-algebra.

7. Let $b \in B$ be an element integral over A. Then, it has a monic polynomial $f(x) = x^n + \sum_{j=1}^{n-1} a_j x^j$ such that $f(b) = 0$ and $a_j \in A$. Since $f(b) = 0$ and $a_0 \in A$, $b^n + a_{n-1}b^{n-1} \cdots + a_1 b \in A$. Since $B \setminus A$ is multiplicatively closed, either $b \in A$ or $b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1 \in A$. If $b \in A$, done, otherwise, since $a_1 \in A$ thus $b(b^{n-2} + \cdots + a_2) \in A$ thus either b or $b^{n-2} + \cdots + a_2 \in A$. Iterating this process, we can arrive that $b + a_{n-1} \in A$, thus $b \in A$.

8. (a) Let K_B be a field of fraction of B, let Ω be an algebraic closure of K_B. Then, $f, g \in K_B[x]$ split into linear factors on $\Omega[x]$, say $f = \prod_i (x - \xi_i), g = \prod_j (x - \eta_j)$. Then, each ξ_i and η_j are integral over C since they have a monic polynomial $fg \in C[x]$. Thus, by Viete's theorem, each coefficients of f and g is integral over C. Also, by the transitivity of integral dependence, all coefficients of f and g are integral over A. Since these coefficients are in B, thus they lie in C since C is integral closure of A. Hence, $f, g \in C[x]$.

   (b) For the second claim, all we need to do is to show that there exists an extension ring D containing B containing all ξ_i and η_j so that f, g split into linear factors in $D[x]$. Then by applying the same argument, we are done.

   To see this, it suffices to show below claim

   Claim XXXII. *For any ring B and $f \in B[x]$ monic, there exists a ring D such that $D \supseteq B$ and f can be factorized as a product of degree one monic polynomial.*

   Proof. Use induction. If $\deg(f) = 1$, done, since every solution is in B. If $\deg(f) > 1$, then let $D_1 = B[x]/(f(x))$ where $(f(x))$ is a principal ideal of $B[x]$ generated by $f(x)$. Then for any $h(x) \in B[x]$, let $\overline{h(x)}$ be a coset in D_1 containing $h(x)$. Then we can embed B onto D_1 by $b \mapsto \overline{b}$. Thus, we can identify B as a subring of D_1. Also, in D_1, $\overline{x}$ is a root of f. Thus, $D_1[t] \to D_1[t]/\overline{(t - \overline{x})}$ is surjective ring homomorphism whose kernel contains $(f(t))$, since $f(t)$ mapped into $\overline{f(t)} = \overline{f(\overline{x})} = f(\overline{x}) = 0$. Since kernel itself is $(t - \overline{x})$, thus there exists $f_1(t)$ such that $f_1(t)(t - \overline{x}) = f(t)$. Hence, $\deg(f_1) < \deg(f)$, by inductive hypothesis, there exists D, an extension of D_1 such that $f_1(x)$ splits into linear factors in $D[x]$. Hence, f as a polynomial in $D[x]$, f splits into linear factors in $D[x]$ and D contains B, done. ∎

9. If $f \in B[x]$ is integral over $A[x]$, then

$$f^m + g_1 f^{m-1} + \cdots + g_m = 0$$

for some $g_i \in A[x], m \in \mathbb{N}$. Let r be an integer larger than m and the degrees of g_i for all i. Let $f_1 = f - x^r$. Then,

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \cdots + g_m = 0.$$

Then by expanding all products, we can say that

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0$$

where $h_m = (x^r)^m + g_1(x^r)^{m-1} + \cdots g_m \in A[x]$. Thus, $f_1^m + h_1 f_1^{m-1} + \cdots + h_{m-1} f_1 = f_1(f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}) \in A[x]$. By Exercise 5.8, this implies that $f_1 \in C[x]$. Hence, $f = f_1 + x^r \in C[x]$.

10. (a) $(a) \to (b)$: As we did in the proof of Exercise 5.1, we can factor out the map f^* as

$$A \xrightarrow{r} f(A) \xrightarrow{s} B \implies \mathrm{Spec}(B) \xrightarrow{s^*} \mathrm{Spec}(f(A)) \xrightarrow{r^*} \mathrm{Spec}(A).$$

Then, f^* closed if and only if of s^* is closed, since r^* is already closed as a homeomorphism (Exercise 1.21 iv)) of $\mathrm{Spec}(f(A)) \cong V(\ker(f))$. Thus, we can just assume that $A \subseteq B$ with $f = i : A \to B$ be an inclusion map.

To see f has going-up property, it suffices to show that if there exists chain of prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ in A and $\mathfrak{q}_1$ is a prime ideal of B such that $\mathfrak{p}_1 = \mathfrak{q}_1 \cap A$, then $\exists \mathfrak{q}_2 \in \mathrm{Spec}(B)$ such that $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$. Notes that in this case we do not assume that B is integral over A. Instead, we assume that $f : A \to B$ an inclusion induces a closed mapping $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Also, this is equivalent to saying that $f^*|_{V(\mathfrak{q}_1)} : V(\mathfrak{q}_1) \to V(\mathfrak{p}_1)$ is surjective map. (Notes that the contraction of any prime ideal containing $\mathfrak{q}_1$ contains $\mathfrak{p}_1$, thus domain and codomain of this map is well-defined.)

Now, since $V(\mathfrak{q}_1)$ is closed, $f^*(V(\mathfrak{q}_1))$ is closed set containing $f^*(q_1) = q_1 \cap A = \mathfrak{p}_1$. Hence, $f^*(V(\mathfrak{q}_1)) \supseteq \overline{\{\mathfrak{p}_1\}}$, a closure of $\{\mathfrak{p}_1\}$. Since $\overline{\{\mathfrak{p}_1\}} = V(\mathfrak{p})$ by Exercise 1.18 ii), $f^*(V(\mathfrak{q}_1)) \supseteq V(\mathfrak{p})$. Thus, the map f^* is surjective.

$(b) \iff (c)$: Notes that $f^*|_{V(\mathfrak{q}_1)} : V(\mathfrak{q}_1) \to V(\mathfrak{p}_1)$ is identified by the map $\overline{f}^* : \mathrm{Spec}(B/\mathfrak{q}_1) \to \mathrm{Spec}(A/\mathfrak{p}_1)$ comes from $\overline{f} : A/\mathfrak{p} \to B/\mathfrak{q}$, by Exercise 3.21, iii). Thus, if (c) holds, then by letting $\mathfrak{p} = \mathfrak{p}_1$ and $\mathfrak{q} = \mathfrak{q}_1$ in (c), $\overline{f}^* : \mathrm{Spec}(B/\mathfrak{q}_1) \to \mathrm{Spec}(A/\mathfrak{p}_1)$ is surjective, thus by Exercise 3.21, iii), $f^*|_{V(\mathfrak{q}_1)} : V(\mathfrak{q}_1) \to V(\mathfrak{p}_1)$ is surjective, hence (a) holds, which implies (b).

Conversely, if (b) holds, then $f^*|_{V(\mathfrak{q}_1)} : V(\mathfrak{q}_1) \to V(\mathfrak{p}_1)$ is surjective as we've seen above for arbitrary choice of $\mathfrak{q}_1$. Thus, by Exercise 3.21 iii), $\overline{f}^* : \mathrm{Spec}(B/\mathfrak{q}_1) \to \mathrm{Spec}(A/\mathfrak{p}_1)$ is surjective for any $\mathfrak{q}_1$ in $\mathrm{Spec}(B)$. This implies (c).

(b) As we did in the proof of Exercise 5.1, we can factor out the map f^* as

$$A \xrightarrow{r} f(A) \xrightarrow{s} B \implies \mathrm{Spec}(B) \xrightarrow{s^*} \mathrm{Spec}(f(A)) \xrightarrow{r^*} \mathrm{Spec}(A).$$

Then, f^* open if and only if of s^* is open, since r^* is already open as a homeomorphism (Exercise 1.21 iv)) of $\mathrm{Spec}(f(A)) \cong V(\ker(f))$. Thus, we can just assume that $A \subseteq B$ with $f = i : A \to B$ be an inclusion map.

We claim that

Claim XXXIII. *If $\mathfrak{p}' \subseteq \mathfrak{p} \in U$ for some Zariski open set U of* $\mathrm{Spec}(A)$*, then $\mathfrak{p}' \in U$.*

Proof. U^c is closed set, thus if $\mathfrak{p}' \in C$ then $\overline{\mathfrak{p}'} \subseteq C$, thus $\overline{\mathfrak{p}'} = V(\mathfrak{p}') \subseteq C$ by Exercise 1.18 ii). Hence, $\mathfrak{p} \in C$, contradiction. □

Now we will show $(a') \implies (c')$ and $(b') \iff (c')$. This is equivalent to saying that $(a') \implies (b') \iff (c')$.

$(a') \to (c')$: By Exercise 3.23, from the construction of presheaf, we know that $\varinjlim_{\mathfrak{q} \in U} B(U) \cong B_{\mathfrak{q}}$, where U is an open sets in the Zariski topology of $\mathrm{Spec}(B)$ containing $\mathfrak{q}$. If we define $S_{\mathfrak{q}} = B - \mathfrak{p}$ as a poset such that $f \leq g$ iff $X_g \subseteq X_f$, then we can redefine presheaf A from topology of $\mathrm{Spec}(B)$ to ring as presheaf from $S_{\mathfrak{q}} = B - \mathfrak{q}$ to ring from the map sending f to X_f. Thus, we can rewrite $B_{\mathfrak{q}}$ as

$$\varinjlim_{t \in S_{\mathfrak{q}}} B_t \cong B_{\mathfrak{q}}.$$

Hence
$$f^*(\mathrm{Spec}(B_{\mathfrak{q}}) \underbrace{=}_{\text{Exercise 3.26}} \bigcap_{t\in S_{\mathfrak{q}}} f^*(\mathrm{Spec}(B_t)) \underbrace{=}_{\text{Exercise 3.21 i)}} \bigcap_{t\in S_{\mathfrak{q}}} f^*(X_f).$$

Since f^* is open map and X_f is an open neighborhoods of $\mathfrak{q}$ for any f, $f^*(X_f)$ is an open neighborhood of $\mathfrak{p} = f^*(\mathfrak{q})$. Hence, by Exercise 3.22,$f^*(X_f)$ contains the canonical image of $\mathrm{Spec}(A_{\mathfrak{p}})$ in $\mathrm{Spec}(A)$ for all f. Thus $f^*(\mathrm{Spec}(B_{\mathfrak{q}})$ contains $\mathrm{Spec}(A_{\mathfrak{p}})$. Conversely, for any $\mathfrak{q}' \in \mathrm{Spec}(B_{\mathfrak{q}}) \subseteq \mathrm{Spec}(B)$, $\mathfrak{q}' \subseteq \mathfrak{q}$ thus $f^*(\mathfrak{q}') \subseteq f^*(\mathfrak{q}) = \mathfrak{p}$, thus $f^*(\mathfrak{q}') \in \mathrm{Spec}(A_{\mathfrak{p}}$ as the canonical image in $\mathrm{Spec}(A)$, by Proposition 3.11 iv). Hence, $f^*(\mathrm{Spec}(B_{\mathfrak{q}}) \subseteq \mathrm{Spec}(A_{\mathfrak{p}})$. Thus, $f^*|_{\mathrm{Spec}(B_{\mathfrak{q}}}$ is surjective map.

$(b') \iff (c')$: Suppose (c') holds. Let $\mathfrak{p}_1 = \mathfrak{q}_1 \cap A$ for some prime ideal $\mathfrak{q}_1$ in B. Then, $f^* : \mathrm{Spec}(B_{\mathfrak{q}_1}) \to \mathrm{Spec}(A_{\mathfrak{p}_1}$ is surjective. Thus, for any $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$, $\mathfrak{p}_2 \in \mathrm{Spec}(A_{\mathfrak{p}_1}$, hence $\exists \mathfrak{q}_2 \in \mathrm{Spec}(B_{\mathfrak{q}_1})$ such that $\mathfrak{q}_2 \cap A = f^*(\mathfrak{q}_2) = \mathfrak{p}_2$. Conversely, suppose (b') holds. Then, for any prime ideal $\mathfrak{q}$ of B, with $\mathfrak{p} = \mathfrak{q}^c$, think a map $f^* : \mathrm{Spec}(B_{\mathfrak{q}}) \to \mathrm{Spec}(A_{\mathfrak{p}})$. First of all, it is well-defined since for any $\mathfrak{q}' \in \mathrm{Spec}(B_{\mathfrak{q}})$, $\mathfrak{q}' \subseteq \mathfrak{q}$, thus $f^*(\mathfrak{q}') \subseteq f^*(\mathfrak{q}) = \mathfrak{p}$, hence $f^*(\mathfrak{q}') \in \mathrm{Spec}(A_{\mathfrak{p}})$. Also, for any $\mathfrak{p}' \subseteq \mathfrak{p}$, $\mathfrak{p}' \in \mathrm{Spec}(A_{\mathfrak{p}})$. Going-down property implies that $\exists \mathfrak{q}' \subseteq \mathfrak{q}$ (which implies $\mathfrak{q}' \in \mathrm{Spec}(B_{\mathfrak{q}})$) such that $\mathfrak{q}' \cap A = \mathfrak{p}' \in \mathrm{Spec}(B_{\mathfrak{q}})$, hence $f^*(\mathfrak{q}') = \mathfrak{p}'$. Thus, f^* is surjective map. Since $\mathfrak{q}$ was arbitrarily chosen, we can conclude that (c') holds.

11. By Exercise 3.18, (c') holds. Thus by Exercise 5.10 ii), (b') holds.

12. A^G is a subring of A, since $1 \in A^G$ (every automorphism preserves 1) and if $a, b \in A^G$, then $ab, a-b \in A^G$ by homomorphic property. Thus, A^G is closed under subtraction and multiplication. (This is criteria for subring.)

    If $x \in A$, notes that $f(t) = \prod_{\sigma\in G}(t - \sigma(x))$ has x as a solution since G has identity morphism. Now all coefficients of $f(t)$ are in A^G, since it is symmetric polynomial of $\sigma(x)$ for all $\sigma \in G$, thus it is in A^G. Also, f is monic. Hence, x is integral over A^G. Since x is arbitrary, done.

    For the second statement, define action of $\sigma \in G$ on $S^{-1}A$ as $\sigma(a/s) = \sigma(a)/\sigma(s)$. It is well-defined since if $a/s = b/t$, then there exists $q \in S$ such that $qta = qbs$. This implies $\sigma(q)\sigma(t)\sigma(s) = \sigma(qta) = \sigma(qbs) = \sigma(q)\sigma(b)\sigma(s)$. Hence $\sigma(a)/\sigma(s) = \sigma(b)/\sigma(t)$. Now let $(S^{-1}A)^G$ be a subring of G-invariants. To see it is isomorphic to $(S^G)^{-1}A^G$, first of all, we can see $(S^G)^{-1}A^G$ as an embedded subring of $S^{-1}A$. Also, notes that for any $a/s \in (S^G)^{-1}A^G$, $\sigma(a/s) = \sigma(a)/\sigma(s) = a/s$, thus $(S^G)^{-1}A^G \subseteq (S^{-1}A)^G$. Lastly, if $a/s \in (S^{-1}A)^G$, then $\sigma(a/s) = a/s$. Now, let $s' := \prod_{\sigma\in G-\{1_G\}} \sigma(s)$. Then, $ss' \in A^G$. Hence, $a/s \cdot ss'/1 = as'/1 \in (S^{-1}A)^G$. Now for each $\sigma \in G$,
    $$\sigma(as'/1) = as'/1$$
    Hence $\exists t_\sigma \in S$ such that
    $$t_\sigma\sigma(as') = t_\sigma as'.$$
    Now let $t = \prod_{\gamma\in G}(\gamma(\prod_{\sigma\in G} t_\sigma))$. This is in S^G since applying any automorphism on t is just permutation of product. Since t_σ contained in t as a divisor,
    $$\sigma(tas') = t\sigma(as') = tas'.$$
    Hence, $tas' \in A^G$. Thus,
    $$a/s = ts'a/ts's \in (S^G)^{-1}A^G,$$
    done,

13. Let $\mathfrak{q}, \mathfrak{q}' \in P$. Then, take $x \in \mathfrak{q}$, $t = \prod_{\sigma\in G} \sigma^{-1}(x)$. Then, $t \in A^G \cap \mathfrak{q} = \mathfrak{p} \subseteq \mathfrak{q}'$ since t is product of x with some others. Since $\mathfrak{q}'$ is prime, at least one $y = \sigma^{-1}(x)$ should lie in $\mathfrak{q}'$. Hence, $\sigma(y) = x$, which implies that $x \in \sigma(\mathfrak{q}')$. Since x was chosen arbitrarily, $\mathfrak{q} \subseteq \bigcup_{\sigma\in G} \sigma(\mathfrak{q}')$. Since σ is autormophism, $\sigma(\mathfrak{q}')$ is still prime ideals for any $\sigma \in G$, thus by Proposition 1.11 i), $\mathfrak{q} \subseteq \gamma(\mathfrak{q}')$ for some γ. Since
    $$\gamma(\mathfrak{q}') \cap A^G = \gamma(\mathfrak{q}') \cap \gamma(A^G) \underbrace{=}_{\text{since } \sigma \text{ is automorphism}} \gamma(\mathfrak{q} \cap A^G) = \gamma(\mathfrak{p}) = \mathfrak{p},$$

Corollary 5.9 says that $\mathfrak{q} = \gamma(\mathfrak{q}')$. Thus, first of all, from the arbitrariness of $\mathfrak{q}$ and $\mathfrak{q}'$ G sends elements in P to another elements in P. And, definitely, 1_G preserves $\mathfrak{q}$, and satisfies group action axioms. (For the composition, notes that acting on P is just action on A, and in this case composition axiom holds.) Finally, G transitively acts on P, i.e., for any two $\mathfrak{q}, \mathfrak{q}' \in P$, there exists $\gamma \in G$ such that $\gamma(\mathfrak{q}) = \gamma(\mathfrak{p}')$, by above result. To see P is finite, notes that just action of G on P has only one orbit; P itself. By the orbit-stabilizer theorem,

$$|P| = |G|/|\mathrm{Stab}(\mathfrak{q})| \leq |G| < \infty.$$

14. For any $b \in B$, b is integral over A, i.e.,

$$\sum_{j=1}^{n} a_j b^j = 0$$

for some $a_j \in A$ with $a_n = 1$. Choose $\sigma \in G$. Then,

$$0 = \sigma(0) = \sum_{j=1}^{n} a_j \sigma(b)^j$$

since σ fixes $K \supseteq A$. Hence, $\sigma(b)$ is integral over A^G, thus $\sigma(b) \in B$. This implies $\sigma(B) \subseteq B$.

Conversely, by replacing σ to σ^{-1} we can get $\sigma^{-1}(B) \subseteq B$. Hence, $B = \sigma(\sigma^{-1}(B)) \subseteq \sigma(B)$. (This comes from the fact that every field homomorphism is injective.) Since σ was arbitrarily chosen, $\sigma(B) = B$ for all $\sigma \in G$.

To see $A = B^G$, notes that

$$B^G = B \cap L^G = B \cap K = A$$

since $B^G \subseteq L^G$ and K is fixed field of G in Galois theory, and B is integral closure of A in K.

15. As hint suggested, we can assume that L is separable over K or purely inseparable over K, since every algebraic extension can be decomposed with a separable extension L_0 over K and purely inseparable extension L over L_0 [2][Theorem 6.6 in V. §6.]. Thus, if we show this Exercise in two cases, then $\mathrm{Spec}(B) \to \mathrm{Spec}(B_0) \to \mathrm{Spec}(A)$ has finite fiber, where B_0 is integral closure of A on L_0.

So suppose that L is separable over K. Then, using[2][Corollary 1.6 in VI. §1.], there exists L' which is finite Galois over K, i.e., finite normal separable extension over K containing L. Let C be an integral closure of A in L'. Then, C contains B. and by transitivity of integral dependence, C is integral over B. Now, let $P = \{\mathfrak{q} \in \mathrm{Spec}(B) : \mathfrak{q} \cap A = \mathfrak{p}\}$. Then, by Theorem 5.10, for each $\mathfrak{q} \in P$, $\exists \mathfrak{q}' \in \mathrm{Spec}(C)$ such that $\mathfrak{q}' \cap B = \mathfrak{q}$. Thus, if we let $P' = \{\mathfrak{q}' \in \mathrm{Spec}(C) : \mathfrak{q}' \cap B \in P\}$, then for any $\mathfrak{q}' \in P'$, $\mathfrak{q}' \cap A = \mathfrak{p}$. However, by letting $G = \mathrm{Gal}(L'/K)$, which is finite group, we know that $A = C^G$ by Exercise 5.14. Thus, by Exercise 5.13, P' is finite. Since $|P'| \geq |P|$ by Theorem 5.10, P is finite. Done.

Suppose L be a purely inseparable over K. By [2][p.249, P. Ins. 3], for any $x \in L$, $x^{p^m} \in K$ for some $m \in \mathbb{N}$ where $p = \operatorname{char} K$. If $\mathfrak{q}^c = \mathfrak{q} \cap A = \mathfrak{p}$, then for any $x \in \mathfrak{q}^c$, $x^{p^m} \in K$. Thus, $x^{p^m} \in \mathfrak{q} \cap K \subseteq B \cap K = A$, since A is integrally closed (in K.) Thus, $x^{p^m} \in \mathfrak{q} \cap A = \mathfrak{p}$. Thus, if we let $\mathfrak{q}' := \{x \in B : x^{p^m} \in \mathfrak{p} \text{ for some } m \in \mathbb{N}\}$, then $\mathfrak{q} \subseteq \mathfrak{q}'$. Conversely, if $x \in \mathfrak{q}'$, then, $x^{p^m} \in \mathfrak{p} \subseteq \mathfrak{q}$ for some $m \in \mathbb{N}$, and since $\mathfrak{q}$ is prime, $x \in \mathfrak{q}$. Thus, $\mathfrak{q} = \mathfrak{q}'$. Now to see a bijection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, define $\mathfrak{q} := \{x \in B : x^{p^m} \in \mathfrak{p} \text{ for some } m \in \mathbb{N}\}$. Then, for any $x, y \in \mathfrak{q}$, $x^{p^m} \in \mathfrak{p}$, $y^{p^n} \in \mathfrak{p}$ for some $m, n \in \mathbb{N}$, thus

$$(x+y)^{p^{\max(m,n)}} = (x^{p^{\max(m,n)}} + y^{p^{\max(m,n)}}) \in \mathfrak{p}.$$

Hence $\mathfrak{q}$ is closed under addition. Also, for any $b \in B$, it has $n \in \mathbb{N}$ such that $b^{p^n} \in K \cap B = A$, thus

$$(bx)^{p^{\max(m,n)}} = b^{p^{\max(m,n)}} x^{p^{\max(m,n)}} \in \mathfrak{p}$$

thus $\mathfrak{q}$ is ideal. To see it is prime, if $xy \in \mathfrak{q}$, with $x^{p^n}, y^{p^m} \in K \cap A = B$ with $(xy)^{p^l} \in \mathfrak{p}$, then take $q = mnl$. Then

$$(xy)^{p^q} = x^{p^q} y^{p^q} \in \mathfrak{p} \text{ with } x^{p^q}, y^{p^q} \in A.$$

Since $\mathfrak{p}$ is prime, either x^{p^q} or $y^{p^q} \in \mathfrak{p}$, thus either x or y in $\mathfrak{q}$ by construction.

Thus, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, there exists $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Hence, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Also, it is injective since if $\mathfrak{q} \cap A = \mathfrak{q}' \cap A = \mathfrak{p}$ for some $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(B)$, then by above argument, both $\mathfrak{q}$ and $\mathfrak{q}'$ contains a prime ideal $\mathfrak{q}'' = \{x \in B : x^{p^n} \in \mathfrak{p} \text{ for some } n \in \mathbb{N}\}$, then by Corollary 5.9, $\mathfrak{q}' = \mathfrak{q}'' = \mathfrak{q}$. This shows that f^* is injective. Thus, f^* is bijectively continuous.

Now we can claim that

Claim XXXIV. *Let A be an integrally closed domain, K its field of fraction and L a finite purely inseparable extension field of K. Let B be the integral closure of A in L. Then, $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induced by the canonical inclusion $f : A \to B$ is homeomorphism.*

Proof. By above argument, we know that f^* is bijectively continuous map. By Exercise 5.1, f^* is closed map. Thus from the topological fact that bijectively continuous map is homeomorphism if and only if it is closed map, f^* is homeomorphism. $\square$

Thus, any fiber is singleton, i.e., finite.

16. This problem want us to show that

Lemma Noether's normalization lemma. *Let k be a field and let $A \neq 0$ be a finitely generated k-algebra. Then there exist elements $y_1, \cdots, y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1, \cdots, y_r]$.*

To prove this, we need to distinguish a case when k is infinite and k is finite.

(a) Suppose k is infinite. By definition of finitely generated algebra, there exists a surjective k-module homomorphism (i.e., linear transformation) $\phi : k[x_1, \cdots, x_n] \to A$ which is onto. (Thus, just identify $\phi(x_i)$ with x_i for economy of notation.) We can renumber the x_i so that $x_1, \cdots, x_r$ are algebraically independent over k and each of $x_{r+1}, \cdots, x_n$ is algebraic over $k[x_1, \cdots, x_r]$. Now proceed induction. If $n = r$, done. Suppose $n > r$ and the result is true for $n-1$ generators. Since x_n is algebraic over $k[x_1, \cdots, x_n]$, there exists a polynomial $f \neq 0$ in n variables such that $f(x_1, \cdots, x_n) = 0$. Let F be the homogeneous part of highest degree in f, i.e, if we let $f = \sum_{j=0}^{M} f_j(x_1, \cdots, x_n)$ where f_j is homogeneous of degree j, then $F := f_M$. Since k is infinite, there exists $\lambda_1, \cdots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \cdots, \lambda_{n-1}, 1) \neq 0$. (To see this, notes that $F = \sum_{j=0}^{M} g_j x_n^j$ for some $g_j \in k[x_1, \cdots, x_{n-1}]$ which is homogeneous of degree $M - j$ respectively. Thus, $G(x_1, \cdots, x_{n-1}) := F(x_1, \cdots, x_{n-1}, 1) = \sum_{j=0}^{M} g_j$ is nonzero if F is nonzero. Therefore, $Z(G) \neq k^{n-1}$, since only polynomial having k^{n-1} as a solution set is zero polynomial, by below Claim XXXV. Thus we can find such lambdas.) Put $x_i' = x_i - \lambda_i x_n$ for $i \in [n-1]$. Then, if we let $F = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = m} c_\alpha x^\alpha$ where $x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}$,

$$F = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = m} c_\alpha x^\alpha = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = m} c_\alpha \prod_{i=1}^{n} x_i^{\alpha_i} = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = m} c_\alpha x_n^{\alpha_n} \prod_{i=1}^{n-1} (x_i' + \lambda_i x_n)^{\alpha_i}.$$

Hence, coefficients of $x_n^{\deg F}$ in F is $F(\lambda_1, \cdots, \lambda_{n-1}, 1) \neq 0$. (To see this, notes that $\lambda_i x_n$ term has a power α_i, so that it gives a form $\prod_{i=1}^{n} x_i^{\alpha_i}$ in F changed with $x_i = \lambda_i$.) Hence, by letting $C = F(\lambda_1, \cdots, \lambda_{n-1}, 1)$ and dividing $f(x_1' + \lambda_1, \cdots, x_{n-1}' + \lambda_{n-1}, x_n)$ by C, we can get a monic polynomial with respect to x_n. (Since F is the highest degree part of f thus dividing by C gives monic term $x_n^{\deg F}$, which is the highest term in $f(x_1' + \lambda_1, \cdots, x_{n-1}' + \lambda_{n-1}, x_n)$. Thus, $f(x_1' + \lambda_1, \cdots, x_{n-1}' + \lambda_{n-1}, x_n) = 0$ (since we set $x_i = x_i' - \lambda_i x_n$), and we can think $f(x_1' + \lambda_1, \cdots, x_{n-1}' + \lambda_{n-1}, x_n)$ as an element in $k[x_1', \cdots, x_{n-1}'][x_n]$. Hence, x_n is integral over $A' := k[x_1', \cdots, x_{n-1}']$. This implies that A is integral over $A'[x_n]$, therefore by transitivity of integral dependence, A is integral over A'. Hence by the inductive hypothesis, A' has an elements $y_1, \cdots, y_r \in A'$ which are algebraically independent over k and such that A' is integral over $k[y_1, \cdots, y_r]$. Since A is integral over A', hence by the transitivity of integral dependence, A is integral over $k[y_1, \cdots, y_r]$. Done.

Claim XXXV. *A polynomial function $g : k^n \to k$ is zero if and only if $g = 0 \in k[x_1, \cdots, x_n]$*

Proof. If $n = 1$, done. Suppose it holds for $n-1$ and $g : k^n \to k$ be a polynomial having kernel k^n. Then, $g \in k[x_1, \cdots, x_{n-1}][x_n]$, thus $g = \sum_{j=1}^m g_j x_n^j$ for some polynomials $g_j \in k[x_1, \cdots, x_{n-1}]$. Then for any $a \in k$, $g(x_1, \cdots, x_{n-1}, 0) = g_0$ is polynomial in $k[x_1, \cdots, x_{n-1}]$ sending all k^{n-1} to 0, thus $g_0 = 0$ by inductive hypothesis. Now, think g/x_n, which is still polynomial and whose constant part is g_1. Then, for any $a \neq 0$, $g(x_1, \cdots, x_{n-1}, a)/a = 0/a = 0$ implies $g_1(x_1, \cdots, x_{n-1}) = 0$ for all k^{n-1}. Thus $g_1 = 0$ by inductive hypothesis. By doing this argument iteratively, we can get $g = g_m x_n^m$, thus $0 = (g/x_n^m)(x_1, \cdots, x_{n-1}, a) = g_m(x_1, \cdots x_{n-1})$ for any k^{n-1} implies $g_m = 0$, hence $g = 0$. $\square$

(b) We refer proof of [5]. Suppose k is finite. Then, by definition of finitely generated algebra, there exists a surjective k-module homomorphism (i.e., linear transformation) $\phi : k[x_1, \cdots, x_n] \to A$ which is onto. (Thus, just identify $\phi(x_i)$ with x_i for economy of notation.) We can renumber the x_i so that $x_1, \cdots, x_r$ are algebraically independent over k and each of $x_{r+1}, \cdots, x_n$ is algebraic over $k[x_1, \cdots, x_r]$. Now proceed induction. If $n = r$, done. Suppose $n > r$ and the result is true for $n-1$ generators. Since x_n is algebraic over $k[x_1, \cdots, x_n]$, there exists a polynomial $f \neq 0$ in n variables such that $f(x_1, \cdots, x_n) = 0$.

Let $d > \deg f$, and $x_i' = x_i - x_n^{d^i}$ for $i = 1, \cdots, n-1$. Then, by letting $g(x_1', \cdots, x_{n-1}', x_n) := f(x_1' + x_n^{d^1}, \cdots, x_{n-1}' + x_n^{d^{n-1}}, x_n) = 0$, Then if $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$, then each monomial in g has a form $c_\alpha \prod_{i=1}^{n-1} (x_i' + x_n^{d^i})^{\alpha_i}$. Thus, its pure power term of x_n has an exponential

$$\alpha_n \sum_{i=1}^{n-1} \alpha_i d^i.$$

Since d chosen larger than any α_i, each α has distinct pure power term of x_n. Let β be the exponent such that whose pure power term of x_n is the greatest. Then, dividing g by c_β, we can get a monic polynomial of x_n over $A' = k[x_1', \cdots, x_{n-1}']$. Thus, x_n is integral oveer A', and by inductive hypothesis, A' is integral over $k[y_1, \cdots, y_r]$ for some algebraically independent elements y_i, hence $A'[x_n]$ is integral over $k[y_1, \cdots, y_r]$, and since $A'[x_n] = A$, done.

Notes that the second statement of this lemma only holds for when k is infinite, or, at least, I don't know whether it holds when k is finite.

For the second statement, notes that in case of $r = n-1$, then $y_i = x_i'$ is chosen as a linear combination. By applying this more and more, we can see that all $y_1, \cdots, y_r$ are chosen to be linear combination of $x_1, \cdots, x_n$.

To find out linear projection $\pi : k^n \to k^r$ such that

$$X \xrightarrow{\iota} k^n \xrightarrow{\pi} L \cong k^r$$

is surjective map, we can use Exercise 1.28. It suffices to show that there exists a k-algebra homomorphism $\pi^* : P(k^r) = k[y_1, \cdots, y_r] \to P(k^n) = k[x_1, \cdots, x_n]$ such that

$$k[x_1, \cdots, x_n]/I(X) \xleftarrow{\iota^\#} k[x_1, \cdots, x_n] \xleftarrow{\pi^*} k[y_1, \cdots, y_r]$$

and its corresponding regular map $X \to k^n \to L$ is surjective.

Now, from the Noether's normalization lemma, we have $t_1, \cdots, t_r$ be algebraically independent elements in A, and an inclusion map $\phi^* : k[y_1, \cdots, y_r] \to A$ by $y_i \mapsto t_i$. Moreover, as we observed above, these t_i is linear combination of $\overline{x_i}$s in A, i.e.,

$$t_i = \sum_{j=1}^n a_{ij} \overline{x_j}.$$

Thus, we can factor through this map ϕ^* by π^* and ι^* by letting

$$\pi^* : k[y_1, \cdots, y_r] \to k[x_1, \cdots, x_n] \text{ by } y_i \mapsto \sum_{j=1}^{n} a_{ij} x_j \text{ for some } a_{ij} \in k.$$

Then, since ι^* is just canonical map for quotient ring, $\iota^* \circ \pi^*$ sends y_i to $\overline{\sum_{j=1}^n a_{ij} x_j} = \sum_{j=1}^n a_{ij} \overline{x_j} = t_i$. Thus, $\phi^* = \iota^* \circ \pi^*$.

Now we can derive π which induces π^* by letting

$$\pi : k^n \to L \text{ by } (v_1, \cdots, v_n) \mapsto (\sum_{j=1}^{n} a_{1j} v_j, \cdots, \sum_{j=1}^{n} a_{rj} v_j).$$

To check that $\pi^* = (\cdot) \circ \pi$, for any polynomial $g(y_1, \cdots, y_r)$,

$$g \circ \pi(t_1, \cdots, t_n) = g(\sum_{j=1}^{n} a_{1j} t_j, \cdots, \sum_{j=1}^{n} a_{rj} t_j) = \pi^*(g).$$

Thus, we can deduce ϕ by $\phi = \pi \circ \iota$. Now, we need to show that this ϕ is surjective.

To see ϕ is surjective, notes that ϕ^* is injection; if $g, g' \in k[y_1, \cdots, y_n]$ has a property that $\phi^*(g) = \phi^*(g')$, then

$$g(t_1, \cdots, t_n) = g'(t_1, \cdots, t_n).$$

If $g \neq g'$, then this induces a nonzero polynomial whose solution is t_is, which contradicts the assumption of algebraic independence of t_is. Hence $g = g'$ in $k[y_1, \cdots, y_r]$.

Now fix $v \in L$. Then, $\{v\}$ is a variety (defined by zero set of $f_v(\vec{y}) := \vec{y} - v$) , thus canonical inclusion ρ_v induces a map $\rho_v^* : P(L) \cong k[y_1, \cdots, y_r] \to P(\{v\}) \cong k$ by sending g to its evaluation at v, i.e., $g(v)$. (This is because $k[y_1, \cdots, y_n]/(y_1 - v_1, \cdots, y_r - v_r)$ is a field and its canonical map is evaluation.) From the injection ϕ^*, we can think $k[y_1, \cdots, y_r]$ is a subring of A (as $k[t_1, \cdots, t_r]$) such that A is integral over $k[y_1, \cdots, y_r]$. Then define

$$\rho_v^{*,'} : k[t_1, \cdots, t_r] \subseteq A \xrightarrow{(\phi^*)^{-1}} k[y_1, \cdots, y_r] \xrightarrow{\rho_v^*} k.$$

Since k is algebraically closed, $\rho_v^{*,'}$ extends to a homomorphism $\tilde{\rho_v^{*,'}} : A \to k$ by Exercise 5.2. Hence by Exercise 1.28, 1-1 correspondence gives a function $\tilde{\rho_v} : \{v\} \to X$. From

$$\rho_v^{*,'} = \rho_v^* \circ (\phi^*)^{-1},$$

$$\rho_v^{*,'} \circ \phi^* = \rho_v^* \implies \tilde{\rho_v^{*,'}} \circ \phi^* = \rho_v^*.$$

Hence,

$$\phi \circ \tilde{\rho_v} = \rho_v.$$

This means that for any $v \in L$, the canonical injection $\{v\} \to L$ factor through X, i.e., $\{v\} \to L$ is equal to $\{v\} \to X \to L$. Now to see ϕ is surjective, pick $v \in L$. Then, by this factor through map, $\tilde{\rho_v}(v) \in X$ such that $\phi(\tilde{\rho_v}(v)) = v$. Hence, ϕ is onto map.

17. To see that X is not empty, let $A = k[t_1, \cdots, t_n]/I(X)$ be the coordinate ring of X. Then, $A \neq 0$, hence by Exercise 5.16 there exists a linear subspace L of dimension greater than 0 in k^n and a mapping of X onto L. Hence $X \neq \emptyset$. The author call it weak form of Nullstellensatz.

For the second statement, let $\mathfrak{m}$ be a maximal ideal. Then, $A = k[t_1, \cdots, t_n]/\mathfrak{m}$ is not only field, but also finitely generated k-algebra. Thus, by Noether normalization lemma, A has algebraically independent elements $y_1, \cdots, y_r$ such that A is integral over a subalgebra $k[y_1, \cdots, y_r]$. However, since A is field, thus if $r > 1$, then any two elements are algebraically dependent. Hence, just pick $f \in k[t_1, \cdots, t_n]$ such that $\overline{f} = 1$, and let $y_1 := \overline{f}$. Then, the lemma says that A is integral over $k[\overline{f}] = k$. Also notes that

A is a field containing k. But since k is algebraically closed, any algebraic extension of k is contained in itself, thus A is also contained in k. This implies $k = A$. Thus, we have a canonical projection map $\pi : k[t_1, \cdots, t_n] \to A \cong k$. Then, let $a_i = \pi(t_i)$. Then, $(t_1 - a_1, \cdots, t_n - a_n) \subseteq \ker \pi = \mathfrak{m}$. If we show $(t_1 - a_1, \cdots, t_n - a_n)$ is maximal, then done.

Claim XXXVI. *For any field k, an ideal of form $(t_1 - a_1, \cdots, t_n - a_n)$ in $k[t_1, \cdots, t_n]$ is maximal.*

Proof. Let $\phi : k[t_1, \cdots, t_n]$ by $t_i \mapsto t_i + a_i$. Then, ϕ is well-defined endomorphism since it sends 1 to 1 and additivity and multiplicativity holds by construction. Also, it is automorphism since $\phi(x^\alpha) = x^\alpha +$ some other terms when $\alpha \neq 0 \in \mathbb{N}^n$. Thus, $(t_1 - a_1, \cdots, t_n - a_n)$ is maximal if and only if $\phi((t_1 - a_1, \cdots, t_n - a_n)) = (t_1, \cdots, t_n)$ is maximal. But it is maximal since by a map $k[t_1, \cdots, t_n] \to k$ sending $f \mapsto f(0, \cdots, 0)$, its kernel contains $(x_1, \cdots, x_n)$ and if $f \in \ker$, then f has zero constant, thus $f = 0$ or f consists of degree at least 1 monomials, which are elements in $(x_1, \cdots, x_n)$. $\square$

18. This is called Zariski's lemma.

Let $x_1, \cdots, x_n$ generate B as a k-algebra. If $n = 1$, then x_1 has an inverse in B, thus x_1^{-1} can be denoted as a polynomial of x_1, i.e., $x_1^{-1} = \sum_{j=1}^m c_j x_1^j$. Thus, $\left(\sum_{j=1}^m c_j x_1^{j+1}\right) - 1 = 0$. Hence, x_j is a solution of a polynomial in $A[x]$, thus it is algebraic over A.

So assume $n > 1$. Let $A = k[x_1]$ and $K = k(x_1)$ be the field of fraction of A. By the inductive hypothesis, B can be regarded as a finitely generated K-algebra with $n-1$ case, so B is a finite algebraic extension of K. Hence each of $x_2, \cdots, x_n$ satisfies a monic polynomial equation with coefficients in K, i.e., coefficients of the form a/b where $a, b \in A$. If f is the product of denominators of all these coefficients, then each of $x_2, \cdots, x_m$ is integral over A_f. Hence B is integral over A_f since any element in B is generated by $x_1, x_2, \cdots, x_n$. Since $K \subseteq B$, K is also integral over A_f.

Now suppose that x_1 is transcendental over k. Then, A is integrally closed since A is UFD and below claim.

Claim XXXVII. *If A is UFD, then it is integrally closed, i.e., its integral closure in its field of fraction is itself.*

Proof. Any elements in the field of fraction of A can be denoted as a/b where there is no irreducibles of A dividing both a and b simultaneously. Thus, if a/b are integral over A, then by multiplying b^n we can get kind of equation

$$a^n + c_1 a^{n-1} b + \cdots + c_n b^n = 0.$$

Thus, $a^n = -(c_1 a^{n-1} b + \cdots + c_n b^n)$ is divisible by b, hence a^n is divisible by b. Since no irreducible of A divides both a and b, this implies that b is unit in A. Hence, $a/b = ab^{-1} \in A$. $\square$

Hence, A_f is integrally closed by Proposition 5.12, therefore $A_f = K_f \cong K$ since K is integral over A. However, it is not possible by below lemma.

Lemma N. *o polynomial ring $k[x]$ has an element $f \in k[x]$ such that $k[x]_f$ is a nontrivial field.*

Proof. If it has such f. Then, $f \notin k$, since $a/f^m \in k[x]_f$ is equal to $af^{-m}/1$, thus $k[x]_f \cong k[x]$. Thus, $\deg f > 0$, which implies $1 - f \neq 0$. Since $k[x]_f$ is a field, $(1-f)^{-1}$ is in $k[x]_f$, hence $(1-f)^{-1} = g/f^m$ for some $m \in \mathbb{N}$. This implies that $f^m = (1-f)g$ in $k[x]$ Then, in a ring $k[x]/(1-f)$,

$$1 \equiv f \equiv f^m \equiv (1-f)g \equiv 0 \text{ modulo } (1-f)$$

Thus, $k[x]_f/(1-f) \cong 0$, which implies that $1 - f$ is unit in $k[x]$. However, since $\deg(1-f) > 0$, its leading coefficient should be nilpotent by Exercise 1.2 i). However k contains no nilpotent other than 0, contradiction. $\square$

Thus, x_1 is algebraic over k. Hence $A = k[x_1]$ is integral over k, thus K and B are integral over k by transitivity of integral dependence. Since k is field, it means that K and B are finite extension of k.

19. Given k be algebraically closed field, let $\mathfrak{m}$ be a maximal ideal in $k[t_1, \cdots, t_n]$. Let $B = k[t_1, \cdots, t_n]/\mathfrak{m}$. Then, B is both a finitely generated k-algebra and a field. By Exercise 5.18, B is a finite algebraic extension of k. Since k is algebraically closed, $B \cong k$. Thus, let $a_i = \overline{t_i}$. Then, $t_i - a_i \in \mathfrak{m}$ for all i, this implies $\mathfrak{m}$ contains $(t_1 - a_1, \cdots, t_n - a_n)$. By Claim XXXVI, it is maximal, thus $\mathfrak{m} = (t_1 - a_1, \cdots, t_n - a_n)$. Hence, every maximal ideal in the given ring is of the form $(t_1 - a_1, \cdots, t_n - a_n)$.

20. Before starting this, we observe a very basic facts in localization.

    Claim XXXVIII. *If B is a finite (or finitely generated, resp.) A-algebra, and S is a multiplicative subset of A, then $S^{-1}B$ is a finite (or finitely generated, resp.) $S^{-1}A$-algebra.*

    Proof. If B is finite A-algebra, then B is finitely generated as an A-module, thus B has a set of generators $\{b_1, \cdots, b_n\}$. Then, any elements in b can be denoted as $b = \sum_{j=1}^n a_i b_i$ for some $a_i \in A$. Thus, any element in $S^{-1}B$ can be denoted as $b/s = \sum_{j=1}^n a_i b_i/s = \sum_{j=1}^n (a_i/s)(b_i/1)$. Thus, $S^{-1}B$ is also finitely generated as an $S^{-1}A$-module.

    Similarly, if B is finitely generated A-algebra, then there exists $x_1, \cdots, x_n$ such that every elements b in B can be written as polynomial of $x_1, \cdots, x_n$ over $f(A)$ where $f : A \to B$ be a ring map defining algebra structure. Thus, if $b = f(x_1, \cdots, x_n) = \sum_{\alpha \in \mathbb{N}^n} f(a_\alpha) x^\alpha$, then $b/f(s) = \sum_{\alpha \in \mathbb{N}^n} f(a_\alpha)/f(s)(x^\alpha/1)$, thus it is polynomial of $x_1/1, \cdots, x_n/1$ over $\overline{f}(S^{-1}A)$. Thus, $S^{-1}B$ is a finitely generated $S^{-1}A$-algebra. ☐

    Let $S = A - \{0\}$ and let $K = S^{-1}A$, the field of fraction of A. Then, $S^{-1}B$ is a finitely generated K-algebra by above claim. Thus, by Exercise 5.16, there exists $y_1/s_1, \cdots, y_n/s_n$ in $S^{-1}B$ algebraically independent over K and such that $S^{-1}B$ is integral over $K[y_1/s_1, \cdots, y_n/s_n]$. Let $z_1, \cdots, z_m$ generate B as an A-algebra. (Such z_is exist since B is finitely generated A-algebra.) Then, each $z_j/1 \in S^{-1}B$ is integral over $K[y_1/s_1, \cdots, y_n/s_n]$. This implies that there exists a monic polynomial

    $$f_i(x) := \sum_{j=1}^{l} a_{ij} x^j$$

    such that $f_i(z_i/1) = 0$. where $a_{ij} \in K[y_1/s_1, \cdots, y_n/s_n]$. Now, notes that a_{ij} is of form $a_{ij} = \sum_{\alpha \in \mathbb{N}^n} c_{ij} x^\alpha / s^\alpha$. Thus, let $den(a_{ij}) = \prod_{\alpha \in \mathbb{N}^n} s^\alpha$, and let $s = \prod_{i,j} den(a_{ij})$. Then, $sa_{ij} = \sum_{\alpha \in \mathbb{N}^n} (sc_{ij}/s^\alpha) x^\alpha$ with $sc_{ij}/s^\alpha \in A$. Thus, $sa_{ij} \in B' := A[y_1, \cdots, y_n]$ for any i, j. (Notes that $y_1, \cdots, y_n$ is algebraically independent, otherwise there is some polynomial of $y_1, \cdots, y_n$ which equals zero, so by replacing y_i to $s_i \cdot y_i/s_i$, we can get $y_1/s_1, \cdots, y_n/s_n$ are not algebraically independent, contradiction.) Thus, $sf_i \in B'[x]$, which implies that $sf_i \in B'_s[x]$, therefore $f_i \in B'_s[x]$ for each i, where $B'_s = \{1, s, s^2, \cdots\}^{-1} B' = \{1, s, s^2, \cdots\}^{-1} A[y_1, \cdots, y_n]$. This shows that $z_i/1$ is integral over B'_s. Since B is generated by z_i over A, $B_s = \{1, s, s^2, \cdots\}^{-1} B$ is generated by $\{z_i/1\}_{i=1}^m$ over $\{1, s, s^2, \cdots\}^{-1} A$ over $S^{-1}A$. Also, since $\{z_i/1\}_{i=1}^m$ is integral over B'_s and B_s contains B'_s (since y_i are in B, thus it can be embedded into $y_i/1$) this implies that B_s is integral over B'_s.

21. Use the same notation as in Exercise 5.20. Notes that $s \neq 0$ since $s \in A \setminus \{0\}$. Thus, by universal property of localization (Proposition 3.1), f extends to the unique map $f_s : A_s \to \Omega$. Now, $B'_s = A_s[y_1, \cdots, y_n]$. Thus, by sending y_i to 0, we can extend f_s to $f'_s : B'_s \to \Omega$. (It is trivially ring homomorphism.) Now, since B_s is integral over B'_s, by Exercise 5.2, we can extend f'_s to $g : B_s \to \Omega$. Now using the canonical map $h : B \to B_s$, we can extend it to $\phi : B \to \Omega$ by $\phi = g \circ h$.

22. Use the same notation as in Exercise 5.20. Let $v \neq 0$ be an element of B. It suffices to show that there is a maximal ideal of B which does not contain v. Notes that $B_v = B[1/v]$ is a finitely generated B-algebra, and B is finitely generated A-algebra, thus B_v is finitely generated A algebra. (Just as polynomial ring over polynomial ring is polynomial ring.) Thus B_v contains A as an action of $a \in A$ on $1 \in B_v$. Thus, by applying Exercise 5.21 to the ring B_v and its subring A, we obtain $s \neq 0 \in A$ such that if Ω is an algebriacally closed field and $f : A \to \Omega$ is homomorphism for which $f(s) \neq 0$, then f can be extended to a homomorphism $B_v \to \Omega$. Now let $\mathfrak{m}$ be a maximal ideal of A such that $s \neq \mathfrak{m}$. (This is possible since Jacobson radical of A is zero, thus intersection of all maximal ideals is zero.) Let $k = A/\mathfrak{m}$. Then, the canonical map $f : A \to k$ extends to a map $f : A \to \Omega$, such that $f(s) \neq 0$.

Hence, f extends to a homomorphism $g : B_v \to \Omega$. Now, since $v/1$ is unit in B_v, $g(v/1)$ is also unit in Ω, thus $g(v/1) \neq 0$.

To see $\ker(g) \cap B$ is maximal, notes that $\ker(g) \cap A = \mathfrak{m}$ is maximal ideal of A. Thus,

$$k = A/\mathfrak{m} \leq B_v/\ker(g) \leq \Omega.$$

By below claim, we can see that $B_v/\ker(g)$ is a field. Hence $\ker(g)$ is a maximal ideal in B_v. Hence, its contraction $\ker(g) \cap B$ is a maximal ideal in B.

Claim XXXIX. *Let k be a field and Ω be an algebraic extension of k. Then any subring E containing k and contained in Ω is field.*

Proof. Let $b \neq 0 \in E$. Then, $b^{-1} \in \Omega$. Since Ω is an algebraic extension of k, there is a polyomial $f \in k[x]$ such that $f(b^{-1}) = \sum_{j=1}^{n} a_j b^{-j} = 0$ with $a_j \in k$. (Also since k is a field, we can assume that f is monic.) Now, multiply b^{n-1} to get $b^{-1} = -\sum_{j=1}^{n-1} a_j b^{n-1-j} \in E$. Hence, every nonzero elements in E is unit. So E is a field. □

23. *i)* $\implies$ *ii)*: Let $f : A \to B$ be a ring homomorphism. Then $f(A)$ is a homomorphic image, thus f induces a map $A \to f(A)$. Thus, assume that $B = f(A)$, i.e., f is surjective homomorphism. Then, $B \cong A/\ker f$, thus by Proposition 1.1, every prime ideal of B is extension of prime ideal in A containing $\ker f$. Hence, the contraction of nilradical is an intersection of all prime ideals in A containing $\ker(f)$. By i), this is the same as an intersection of all maximal ideals containing $\ker(f)$. (One direction is trivial. For other direction, let x be an element of intersection of all maximal ideals. If $x \in \ker(f)$, done. Otherwise, x is in every maximal ideal containing $\ker(f)$. Thus for any prime ideal containing $\ker(f)$, it is intersection of some maximal ideals containing $\ker(f)$, hence x is in the prime ideal. Done.) Since an intersection of all maximal ideals containing $\ker(f)$ is contraction of Jacobson radical of B, this implies that Jacobson radical is equal to nilradical, by 1-1 correspondence.

    ii) $\implies$ *iii)*: If $\mathfrak{p}$ is a prime not maximal ideal, then (0) is not a maximal ideal in $A/\mathfrak{p}$ by 1-1 correspondence. However, since $A/\mathfrak{p}$ is integral domain, (0) is nilradical. By ii), (0) is the Jacobson radical. Hence, by 1-1 correspondence, $\mathfrak{p}$, which i s the contraction of (0), is equal to an intersection, say $\bigcap \mathfrak{m}$, of all maximal ideals in A containing $\mathfrak{p}$. Thus for any $\mathfrak{q}$ a prime ideal strictly containing $\mathfrak{p}$, intersect $\mathfrak{q}$ with the $\bigcap \mathfrak{m}$ is just $\mathfrak{p}$. Hence, we can rewrite it as intersection of all prime ideals which contain $\mathfrak{p}$ strictly.

    iii) $\implies$ *i)*: Notes that hints actually denote this way. Suppose *i)* is false. Then, there is a prime ideal which is not an intersection of maximal ideals, say $\mathfrak{p}$. Thus, zero ideal in $A/\mathfrak{p}$ is not Jacboson radical. (If it was, then by 1-1 correspondence $\mathfrak{p}$ is an intersection of maximal ideals.) Thus, the Jacobson radical $\mathfrak{R}$ in $A/\mathfrak{p}$ is a nonzero ideal. Let $f \in \mathfrak{R}$ such that $f \neq 0$. Then, $B_f \neq 0$, since f is not a nilpotent. Thus, B_f has a maximal ideal, whose contraction is a prime ideal $\overline{\mathfrak{q}}$ in $A/\mathfrak{p}$ such that $f \notin \overline{\mathfrak{q}}$. Also notes that any prime ideal containing $\overline{\mathfrak{q}}$ strictly contains f, since otherwise that prime ideal should be a proper ideal in B_f by Proposition 3.14's 1-1 correspondence of prime ideal. Hence, by iii) (which is applicable for its homomorphic image by Proposition 1.1), $\overline{\mathfrak{q}}$ is an intersection of prime ideals strictly containing $\overline{\mathfrak{q}}$, and since every strictly greater prime ideals contains f, this implies $f \in \overline{\mathfrak{q}}$, contradicting the fact that $f \notin \overline{\mathfrak{q}}$.

    Notes that

    Claim XL. *Every homomorphic image of Jacobson ring is Jacobson ring.*

    Proof. Let B be a homomorphic image of A. By Exercise 1.21 iv), for any prime ideal $\mathfrak{p}$ of B, $\operatorname{Spec}(B/\mathfrak{p}) \cong V(\ker(f))$ where $f : A \to B \to B_{\mathfrak{p}}$ be a surjective ring homomorphism making $B_{\mathfrak{p}}$ be an A-algebra. Then, by ii), (0) is both nilradical and Jacobson radical. Hence, $\mathfrak{p}$ is intersection of all maximal ideals containing $\mathfrak{p}$ in B. Since $\mathfrak{p}$ chosen arbitarily, B satisfies condition i). □

    Also, there is another equivalent definition from EGA, according to [6].

Claim XLI. *For a ring A, the followings are equivalent.*

(a) *For any radical ideal $\mathfrak{a}$, we have $\bigcap_{\substack{\mathfrak{m}\in\max(A)\\ \mathfrak{m}\supseteq\mathfrak{a}}}\mathfrak{m} = \mathfrak{a}$.*

(b) *For all prime ideal $\mathfrak{p}$, we have $\bigcap_{\substack{\mathfrak{m}\in\max(A)\\ \mathfrak{m}\supseteq\mathfrak{p}}}\mathfrak{m} = \mathfrak{p}$.*

(c) *For every integral domain A' which is homomorphic image of A, $\bigcap_{\mathfrak{m}\in\max(A')}\mathfrak{m} = 0$.*

Proof. i) implies ii) is obvious since any prime is radical. Also, since any radical ideal is intersection of prime ideals containing it, so ii) implies i). For iii), $A' = A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$, thus by Proposition 1.1, order preserving 1-1 correspondence, ii) implies iii). Also, if iii) holds, then for any $\mathfrak{p}$ prime ideal of A, it can be rewritten as intersection of maximal ideals in A containing $\mathfrak{p}$, thus iii) implies ii). $\square$

Since condition ii) is just the same as condition of Jacobson ring, these are another definitions.

24. (ii): Let $\mathfrak{q}$ be a prime ideal in B. Then, $B/\mathfrak{q}$ is an integral domain and finitely generated as $A/\mathfrak{q}^c$-algebra. Since A is Jacobson ring, by Exercise 5.23 ii), (0) is both nilradical and Jacobson radical. Thus by Exercise 5.22, $B/\mathfrak{q}$ has zero Jacobson radical. Hence, $\mathfrak{q}$, which is contraction of (0) is the intersection of all maximal ideals containing $\mathfrak{q}$.

    (i): This proof came from [6]. Let $f : A \to B$ be a ring integral homomorphism. Let B' be an integral domain which is homomorphic image of B, and let A' be the image of A in B'. Then, $A' \to B'$ is integral injective homomorphism between two integral domain. Let $\mathfrak{a} = \bigcap_{\mathfrak{m}'\in\max(B')}\mathfrak{m}'$ where $\max(B')$ is a set of all maximal ideals of B'. Then,

$$\mathfrak{a}\cap A' = (\bigcap_{\mathfrak{m}'\in\max(B')}\mathfrak{m}')\cap A' = \bigcap_{\mathfrak{m}'\in\max(B')}(\mathfrak{m}'\cap A').$$

By Corollary 5.8, $\bigcap_{\mathfrak{m}'\in\max(B')}(\mathfrak{m}'\cap A') = \bigcap_{\mathfrak{m}\in\max(A')}\mathfrak{m}$. To see this, we claim that

Claim XLII. *If $A\subseteq B$ is integral domain and B is integral over A, then there exists 1-1 correspondence between* $\max(B)$ *and* $\max(A)$.

Proof. It is clear that for any $\mathfrak{m}\in\max(B)$, $\mathfrak{m}^c = \mathfrak{m}\cap A$ is maximal by Corollary 5.8. Conversely, suppose that $\mathfrak{m}\in\max(A)$. Then, by Theorem 5.10, there exists a prime $\mathfrak{q}\in\operatorname{Spec}(B)$ such that $\mathfrak{q}\cap A = \mathfrak{m}$. Thus, it suffices to show that $\mathfrak{q}$ is maximal. Suppose it is not maximal; then there exists a maximal ideal $\mathfrak{q}'$ strictly containing $\mathfrak{q}$. Hence, $\mathfrak{q}'\cap A = (\mathfrak{q}')^c$ is not only prime but maximal by 1-1 correspondence, and since $\mathfrak{q}'$ contains $\mathfrak{q}$, $\mathfrak{q}'\cap A$ contains $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal, thus $\mathfrak{q}'\cap A = \mathfrak{q}\cap A$. By Corollary 5.9, $\mathfrak{q}' = \mathfrak{q}$. Thus, $\mathfrak{q}$ is maximal. $\square$

Hence, by condition ii) of Jacobson ring, $\mathfrak{a}\cap A' = 0$. This implies $\mathfrak{a} = 0$ by the below claim. Since B' was arbitrarily chosen, B is also Jacobson ring by iii) condition of the above claim.

Claim XLIII. *Let $A\subseteq B$ be an integral domains, B is integral over A. Then for any nonzero ideal $\mathfrak{b}$ of B, $\mathfrak{b}^c = A\cap\mathfrak{b}$ is nonzero.*

Proof. Suppose that $A\subseteq B$. Since $\mathfrak{b}$ is nonzero, there exists $b\in B\setminus A$. Since b is integral over A,

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0.$$

for some $a_i\in A$. Since $b^n + a_{n-1}b^{n-1} + \cdots + a_1 b\in\mathfrak{b}$, $a_0\in\mathfrak{b}$, hence $\mathfrak{b}\cap A\neq\emptyset$. $\square$

A ring is finitely generated if it is finitely generated as $\mathbb{Z}$-algebra. Thus, it suffices to show that $\mathbb{Z}$ and all fields are Jacobson ring. In case of field, it is trivial since all prime ideal is 0 and maximal. Also, $\mathbb{Z}$ has prime ideals either maximal or (0), which is intersection of all prime ideal which is nonzero. Hence, $\mathbb{Z}$ satisfies condition iii) of Jacobson radical.

25. $i) \implies ii)$: Let A' be a image of A in B. Then, A' is Jacobson ring by claim XL, and A' is integral domain as a subring of a field B. Hence, nilradical is equal to 0 and which is equal to Jacobson radical by second condition of Jacobson ring. Now by apply Exercise 5.21 we can find $s \neq 0 \in A'$ satisfying conditions in Exercise 5.21. Now, since the Jacobson radical is zero, we can find a maximal ideal $\mathfrak{m}$ of A' not containing s. Then, $A'/\mathfrak{m}$ is a field, so take Ω be a algebraic closure of $A/\mathfrak{m}$. Then, $f : A' \to A'/\mathfrak{m} \to \Omega$ is a homomorphism where $f(s) \neq 0$, thus by Exercise 5.21, f can be extended to $g : B \to \Omega$. Since B is a field, g is injective, thus, $g(B) \cong B$. Since B is finitely generated over A', there exists $x_1, \cdots, x_n \in B$ such that every elements in B can be written as a polynomial of x_is with coefficients from A'. Now, $g(B)$ is generated by $g(x_i)$ for $i = 1, \cdots, n$. Now, since $g(B) \subseteq \Omega$, $g(x_i)$ is algebraic over $A/\mathfrak{m}$, thus $g(B)$ is finite extension of $A'/\mathfrak{m}$. Thus, it is a finite dimensional $A'/\mathfrak{m}$-vector space, therefore it has a basis. This implies that it is finitely generated $A'/\mathfrak{m}$-module, and using the surjective homomorphism $A \to A' \to A'/\mathfrak{m}$ it is a finitely generated A-module. Hence $B \cong g(B)$ is a finite A-algebra.

$ii) \implies i)$. Let $\mathfrak{p}$ be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. Let f be a nonzero element of B. Then, B_f is finitely generated B-algebra since every elements in B_f is polynomial of $1/f$ with coefficients from B. Thus, using a surjective homomorphism $A \to B = A/\mathfrak{p}$, B_f is also a finitely generated A-algebra. If B_f is a field, then it is finite over A by assumption ii), i.e., B_f is finite A-algebra. Since A acts on B_f through $A \to A/\mathfrak{p} = B$, B_f is also a finite B-algebra. Thus, B_f is integral over B by [3][p.60], thus B is a field by Proposition 5.7, which is not true since $\mathfrak{p}$ is not a maximal ideal. Thus B_f is not a field, therefore has a nonzero prime ideal, and whose contraction in B is a nonzero ideal $\mathfrak{p}'$ such that $f \notin \mathfrak{p}'$. Thus, intersection of all nonzero prime ideals in B is zero, thus intersection of the contraction of those prime ideals, i.e., intersection of all prime ideals containing $\mathfrak{p}$ strictly is $\mathfrak{p}$. Hence, A satisfies condition iii), thus it is Jacobson radical.

26. Notes that every closed set and open set is locally closed by intersecting itself with X, which is open.

Also, locally closed is well-defined; if A is open in its closure $\overline{A}$, then by definition $\exists U$ an open set in X such that $A = U \cap \overline{A}$. Conversely, if U is open and C is closed, then notes that $\overline{U \cap C} = \overline{U} \cap C$. (To see this, notes that if K is closed set containing $U \cap C$, then $K \cap (\overline{U} \cap C)$ is closed set contained in K but containing $U \cap C$. Since $K \cap (\overline{U} \cap C)$ is of form $C' \cap C$ where C' is a closed set containing U. Thus,

$$\overline{C \cap U} = \bigcap_{K \text{ closed, containing } C \cap U} K = \bigcap_{C' \text{ closed, containing U}} C' \cap C = C \cap \bigcap_{C' \text{ closed, containing U}} C' = C \cap \overline{U}.$$

Hence, $C \cap U = \overline{C \cap U} \cap U = \overline{C \cap U} \cap U$, which implies that $C \cap U$ is open in its closure.

(1) $\implies$ (2): For any closed set E in X, take $x \in E$, and take N be a open neighborhood of x. Then, since N is locally closed, thus $N \cap X_0 \neq \emptyset$. Since N was arbitrarily chosen, $x \in \overline{X_0}$. This implies $x \in \overline{E \cap X_0}$. Thus, $E \subseteq \overline{E \cap X_0}$. Conversely, $E \cap X_0 \subseteq E$, hence $\overline{E \cap X_0} \subseteq \overline{E} = E$, done.

(2) $\implies$ (3): This map is just map from topology of X to the topology of X_0 as a subtopology of X. Hence, it is surjective by definition. To see that it is injective, notes that $U \cap X_0 = V \cap X_0$ for some two open sets in X. Then, $U^c \cap X_0 = V^c \cap X_0$, thus $\overline{U^c \cap X_0} = \overline{V^c \cap X_0}$. By ii), $U^c = V^c$, thus $U = V$.

(3) $\implies$ (1): Since the map is bijective, preimage of $\emptyset \cap X_0 = \emptyset$ is just $\emptyset$ in X. Hence, for any nonzero open set U, $U \cap X_0$ is nonzero.

Now we will show equivalence of i), ii) and iii). Define $X_0 = \{\mathfrak{m} \in \mathrm{Spec}(A) : \mathfrak{m} \text{ is maximal.}\}$

$i) \implies ii)$: Notes that arbitrary closed set in $\mathrm{Spec}(A)$ is $V(\mathfrak{a})$ and arbitrary open set is X_f. Thus, let $Y = V(\mathfrak{a}) \cap X_f$, i.e., a prime ideal containing an ideal $\mathfrak{a}$ but not containing an element $f \in A$. (We can assume that $f \notin \mathfrak{a}$, otherwise, $Y = \emptyset$.) Now using the homeomorphism $\phi^* : \mathrm{Spec}(A/\mathfrak{a}) \to V(\mathfrak{a}) \subseteq \mathrm{Spec}(A)$, think about $(\phi^*)^{-1}(Y)$. It is a set of all prime ideals do not containing $\overline{f} \neq 0$. Since A is Jacobson, so is $A/\mathfrak{a}$, thus $\exists \overline{\mathfrak{m}} \in \mathrm{Spec}(A/\mathfrak{a})$ such that $\overline{\mathfrak{m}}$ is a maximal ideal of $\mathrm{Spec}(A/\mathfrak{a})$ not containing $\overline{f}$. Hence, its pullback $\phi^*(\mathfrak{m}) = \mathfrak{m}$. is also a maximal ideal contained in $V(\mathfrak{a})$ but not containing f. Hence, $Y \cap X_0 \neq \emptyset$. Since Y was arbitrary locally closed subset of $\mathrm{Spec}(A)$, done.

$ii) \implies iii)$: Let $Y = V(\mathfrak{a}) \cap X_f$. If Y is singleton, then there is only one prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ but not containing f. Since X_0 is dense by ii), $Y \cap X_0 = Y$, thus $\mathfrak{p}$ is maximal. Hence Y is closed.

$iii) \implies i)$: Suppose A is not a Jacobson ring. Then, $\exists\mathfrak{p}$ which is not a maximal ideal such that $\mathfrak{q} = \bigcap_{\substack{\mathfrak{q}' \in \mathrm{Spec}(A) \\ \mathfrak{q}' \supseteq \mathfrak{p}}} \mathfrak{q}'$ is bigger than $\mathfrak{p}$. Let $f \in \mathfrak{q} \setminus \mathfrak{p}$. Then, $V(\mathfrak{p}) \cap X_f = \{\mathfrak{p}\}$. Thus there is a locally closed singleton which is not closed. Hence iii) fails. So we just showed i) implies iii). Its contrapositive is what we desired to have. .

27. Notes that $K \in \Sigma$, thus Σ is nonempty. Also, if C is a chain in Σ, then let C' be a direct limit of rings in C. Since given map is inclusion and dominate relation also holds by composition of inclusion, and since C' is totally ordered thus it is directed set, we can define C' well as a directed limit using Exercise 2.21. Also, by Exercise 2.22, C' is an integral domain. And, by construction, $C' = \bigcup_{B \in C} B$ by Exercise 2.17.

Now think about a maximal ideal of C'. Since each ring B in C is local ring, it has the maximal ideal $\mathfrak{m}_B$, thus we have an exact sequence

$$0 \to \mathfrak{m}_B \to B \to B/\mathfrak{m}_B \to 0$$

for any $B \in C$. Thus, by Exercise 2.18 and 2.19, we can conclude that

$$0 \to \bigcup_{B \in C} \mathfrak{m}_B \to C' \to C' / \bigcup_{B \in C} \mathfrak{m}_B \to 0$$

is exact. (Notes that direct limit of maximal ideal comes from Exercise 2.17, and $C'/\bigcup_{B\in C} \mathfrak{m}_B \cong \varinjlim B/\mathfrak{m}_B$.) Notes that $\varinjlim B/\mathfrak{m}_B$ is a field, since it is $\bigcup_{B\in C} B/\mathfrak{m}_B$ by Exercise 2.17 (by identifying each ring as a subring of the bigger one) and union of fields in a totally ordered by inclusion is also a field (just easily check all axioms of fields using standard way). Thus, $\bigcup_{B\in C} \mathfrak{m}_B$ is a maximal ideal of C'.

Now we claim that C' is local ring.

Claim XLIV. *A direct limit of local ring is local.*

Proof. Use notation in the problem. Let $(C')^\times$ be set of units in C. We will show that $(C')^\times = C' \setminus \bigcup_{B\in C} \mathfrak{m}_B$. Pick any unit x. Then, there is a ring $B \in C$ such that $x, x^{-1} \in B$. Hence, $x, x^{-1} \notin \mathfrak{m}_B$, thus $x \in C' \setminus \bigcup_{B\in C} \mathfrak{m}_B$. Conversely, pick $x \in \bigcup_{B\in C} \mathfrak{m}_B$. Then, $x \notin \mathfrak{m}_B$ in any subring $B \in C$ containing x. This implies that x is unit in any subring $B \in C$ containing x. Hence, $x^{-1} \in B$ for such B, thus x is unit in C'. Hence, $C' \setminus \bigcup_{B\in C} \mathfrak{m}_B = (C')^\times$. Now Proposition 1.6 assures that C' is local ring. □

Since dominate relation is clear from the construction of maximal ideal of C', $(C', \bigcap_{B\in C} \mathfrak{m}_B)$ is maximal element of C. Thus by Zorn's lemma, Σ has a maximal element.

Now pick A be be such maximal element, and $\mathfrak{m}$ is the maximal ideal. Now let Ω be the algebraic closure of $A/\mathfrak{m}$. Then, $(A, f : A \to A/\mathfrak{m} \to \Omega)$ is inside of Σ in [3][p.65]. If we have $(A', f') \in \Sigma$ of page 65, then $A \subseteq A', f'|_A = f$, thus $\ker(f') \cap A = \ker(f)$, which is maximal ideal in A (since f maps into a field) thus $\ker(f') \cap A = \mathfrak{m}$. Since A' is also a local ring, this implies that $(A', \mathfrak{m}') \geq (A, \mathfrak{m})$ in Σ of this Exercise. ($Notesthat\ker(f') = \mathfrak{m}'$ since f' maps into a field, and A' is local ring.) Thus, by maximality of $(A, \mathfrak{m})$ in the Σ of this Exercise, $A = A', \mathfrak{m} = \mathfrak{m}'$. Hence, $f' = f$, this implies that $(A, f) = (A', f')$, therefore (A, f) is maximal element in Σ of page 65, thus by Theorem 5.21, A is a valuation ring of the field K.

Conversely, let $(A, \mathfrak{m})$ is a valuation ring of K. If it is dominated by other sub local ring of K, say $(A', \mathfrak{m}')$, then by Proposition 5.18, $(A', \mathfrak{m}')$ is a valuation ring of K. Now notes that any element $x \in \mathfrak{m}$ has inverse x^{-1} in $K \setminus A$. Conversely, if $x \in K \setminus A$, then by the definition of valuation ring, $x^{-1} \in \mathfrak{m}$ since x^{-1} is not a unit in A. Thus, $K \setminus A = \{x^{-1} : x \in \mathfrak{m} \setminus \{0\}\}$. Hence,

$$A' \setminus A \subseteq K \setminus A = \{x^{-1} : x \in \mathfrak{m} \setminus \{0\}\} \subseteq \{x^{-1} : x \in \mathfrak{m}' \setminus \{0\}\} = K \setminus A',$$

where $\{x^{-1} : x \in \mathfrak{m} \setminus \{0\}\} \subseteq \{x^{-1} : x \in \mathfrak{m}' \setminus \{0\}\}$ comse from the fact that $\mathfrak{m}' \supseteq \mathfrak{m}$. This implies that $A' \setminus A = (A' \setminus A) \cap (K \setminus A') = \emptyset$. Hence $A = A'$, thus $(A, \mathfrak{m})$ is maximal.

28. (1) $\implies$ (2): Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero ideal. (If one of them is zero, then it is trivial.) If A is a valuation ring, but $\mathfrak{a} \not\subseteq \mathfrak{b}$, then $\exists a \neq 0 \in \mathfrak{a} \setminus \mathfrak{b}$. Now for any $b \neq 0 \in \mathfrak{b}$, either $a/b \in A$ or $b/a \in A$. In case of $a/b \in A$, then $b \cdot a/b = a \in \mathfrak{a} \cap \mathfrak{b}$, contradiction. In case of $b/a \in A$, then $a \cdot b/a = b \in \mathfrak{a} \cap \mathfrak{b}$. Since b was arbitrarily chosen, $\mathfrak{b} \subseteq \mathfrak{a}$.

    (2) $\implies$ (1): For any $a, b \in A$, $(a) \subseteq (b)$ or $(b) \subseteq (a)$. In case of $(a) \subseteq (b)$, $a = xb$ for some $x \in A$, thus $x = a/b$. In case of $(b) \subseteq (a)$, $b = ay$ for some $y \in A$, thus $y = b/a = x^{-1}$. Hence, for x, which is an element of field of fraction of A, i.e., K, either $x \in A$ or $x^{-1} \in A$. Hence, by definition, A is a valuation ring.

    Both $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are still integral domain. Also, in case of $A/\mathfrak{p}$, Proposition 1.1 assures the containment relationship of any two ideals. In case of $A_{\mathfrak{p}}$, if there is two ideals $\mathfrak{a}^e$ and $\mathfrak{b}^e$ in $A_{\mathfrak{p}}$, (we can assume that they are extended from ideal of A by Proposition 3.11), then containment relation also gives containment relation between $\mathfrak{a}^e$ and $\mathfrak{b}^e$ (since generators of an ideal are contained in those of another one.) Thus, both are a valuation ring.

29. A ring B is "local ring of A" means that $B = A_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$. I refer [7]. Let B be a ring containing A and be contained in K. Then by Proposition 5.18 ii), B is a valuation ring of K, and by Proposition 5.18 i), B is a local ring. To see it is a localization of A by some prime ideal $\mathfrak{p}$, let $\mathfrak{m}_B$ be a maximal ideal of B. By the inclusion map, contraction of $\mathfrak{m}_B$ in A, which is $\mathfrak{p} := \mathfrak{m}_B \cap A$ is prime ideal since contraction of prime ideal is prime ideal. Since A is local ring, $\mathfrak{p} \subseteq \mathfrak{m}_A$. Now we claim that $A_{\mathfrak{p}} = B$.

    If $f \in A \setminus \mathfrak{p}$, then $f \notin \mathfrak{m}_B$, thus f is invertible in B, hence for any $a \in A$, $a/f \in B$. Thus, we have an inclusion map $A_{\mathfrak{p}} \to B$. To see other direction, we claim that B dominates $A_{\mathfrak{p}}$ as a subring. From construction, $\mathfrak{m}_B \supseteq \mathfrak{p}A_{\mathfrak{p}}$. By Exercise 5.28 with the fact that A is a valuation ring, $A_{\mathfrak{p}}$ is also a valuation ring. Also notes that field of fraction of $A_{\mathfrak{p}}$ is also K, by the universal property of localization. (Apply Corollary 3.2 iii) on the map $A \to K$). Hence, we can regard $(A_{\mathfrak{p}}, \mathfrak{p}A_{\mathfrak{p}})$ be an element of Σ in Exercise 5.27 such that $(A_{\mathfrak{p}}, \mathfrak{p}A_{\mathfrak{p}}) \leq (B, \mathfrak{m}_B)$. However, Exercise 5.27 says that a set of all valuation rings in Σ is the set of all maximal elements in A. Thus, $A_{\mathfrak{p}} = B$, done.

30. Let $\xi = x + U, \eta = y + U \in \Gamma$. First of all, we need to show that this order is well-defined. If $\xi = x + U = x' + U$ and $\eta = y + U = y' + U$, then by equality, $y(y')^{-1}$ and $y'y^{-1} \in A$, and $x(x')^{-1}, x'x^{-1} \in A$. Thus,

    $$xy^{-1} \in A \iff (x'x^{-1})xy^{-1}(y(y')^{-1}) = x'(y')^{-1} \in A.$$

    Hence, the order is independent of choosing representative.

    Now, to see this order is defined on any two elements, let $\xi = x + U, \eta = y + U \in \Gamma$. Then, since A is valuation ring, either $x \in A$ or $x^{-1} \in A$ (or both). If $x, x^{-1} \in A$, then $x \in U$, thus $\xi = 0 \in \Gamma$, which implies $\eta \geq 0$ since $0y^{-1} = 0 \in A$. So assume that only one of both holds. (Similarly, one of $y \in A$ or $y^{-1} \in A$ holds.) Then,

	$x \in A$	$x^{-1} \in A$
$y \in A$	(1)	(2)
$y^{-1} \in A$	(3)	(4)

    four cases occur. In case of (2) or (3), definitely $yx^{-1} \in A$ or $xy^{-1} \in A$, thus order is determined. In case of (1) or (4), since the inverse of yx^{-1} is xy^{-1}, thus at least one of them should lie in A. Hence, order is still determined. (If both of them lies, then, $\xi = \eta$.)

    Now to see that it is well-defined ordering we need to show it is reflexive, antisymmetry and transitive. For reflexivity, definitely $1 = xx^{-1} \in A$, thus $\xi \geq \xi$ for any $\xi \in \Gamma$. For antisymmetry, if $\xi \geq \eta$ and $\xi \leq \eta$, then $xy^{-1}, yx^{-1} \in A$. This shows that $xy^{-1}, yx^{-1} \in U$. Thus,

    $$\eta = y + U = yxy^{-1} + U = x + U = \xi.$$

    For transitivity, if $\xi \geq \eta \geq \omega$ for each representative x, y, z respectively, then, $xy^{-1}, yz^{-1} \in A$, thus $xy^{-1}yz^{-1} = xz^{-1} \in A$, thus $\xi \geq \omega$. Hence it is well-defined ordering.

Lastly, for the compatibility, if $\xi \geq \eta$, then $xy^{-1} \in A$, thus for some $\omega = z+U$, $\xi\omega = xz+U, \eta\omega = yz+U$, thus, $xz(yz)^{-1} = xy^{-1} \in A$, which implies $\xi\omega \geq \eta\omega$.

Lastly, let $v : K^* \to \Gamma$. Suppose $v(x) \geq v(y)$. Then, $(x+y)y^{-1} = xy^{-1} + 1 \in A$ since $v(x) \geq v(y)$ implies $xy^{-1} \in A$. Hence, $v(x+y) \geq v(y) = \min(v(x), v(y))$. By the similar argument on the case of $v(x) \leq v(y)$, we can conclude that

$$v(x+y) \geq \min(v(x), v(y)).$$

31. I refer [4]. To define a ring well, we need a 0, which is compatible with v. Thus, let $\Gamma = \Gamma \cup \{\infty\}$ which is a monoid such that $\forall \alpha \in \Gamma$, $\alpha + \infty = \infty$ with order $\infty \geq \alpha$. Then,

$$v(0) = v(0 \cdot x) = v(0) + v(x) = \infty + v(x) = \infty.$$

and

$$v(x) = v(0+x) = \min(\infty, v(x)) = v(x).$$

Let $A = \{x \in K^* : v(x) \geq 0\}$. First of all, we claim A is a subring of K. From $v(1) = 0$, (since v is a monoid homomorphism) $1 \in A$. And, $\infty \geq 0$ implies $0 \in A$. Now, $-1 \in A$ since $0 = v(1) = v(-1)+v(-1)$ implies $2v(-1) = 0$. If $v(-1) > 0$ or < 0, then $2v(-1) > 0$ or < 0, thus false. This implies $v(-1) = 0$. Thus, if $x \in K^\times$, then $v(-x) = v(-1 \cdot x) = v(-1) + v(x) = 0 + v(x) = v(x)$. This implies that if $x \in A$, then $-x \in A$. Also, for any $x, y \in A$, $v(x-y) = \min(v(x), v(-y)) = \min(v(x), v(y)) \geq 0$. This implies A is closed under subtraction. Also, $v(xy) = v(x) + v(y) \geq 0$, thus A is closed under multiplication. This implies that A is a subring of K.

Moreover, A is a valuation ring since for any $x \in K^*$, $0 = v(1) = v(x)+v(x^{-1})$ implies $v(x^{-1}) = -v(x)$. Hence if $v(x) = 0$, then $x, x^{-1} \in A$. If $v(x) \neq 0$, then either x or x^{-1} must be in A.

To see that valuation ring induces valuation, let A be a valuation ring of K. Then define $v : K^* \to \Gamma = K^*/A^*$ as the canonical map. By Exercise 5.30, we can show that Γ is the value group of A, thus v is valuation. Now let $A' = \{x \in K^* : v(x) \geq 0\} \cup \{0\}$. Then, $v(x) \geq 0 = v(1)$ iff $x \cdot (1)^{-1} = x \in A$ by construction in Exercise 5.30. Hence, $A' = A$.

Conversely, for given v, we have $\ker(v)$. Thus, $v(K^*) \cong K^*/\ker(v)$. Since $v(x) = 0$ iff $v(x^{-1}) = 0$, thus $\ker(v) = A^*$, a set of all units in A. Hence, $v(K^*) \cong K^*/A^*$. Let $w : K^* \to K^*/A^*$ be a valuation constructed by Exercise 5.30. Then by above argument, $w(x) \geq 0$ iff $x \in A$, hence, $v(x)$ and $w(x)$ has the same sign, i.e., $v(x) \geq 0$ iff $w(x) \geq 0$. Also,

$$v(x) \leq v(y) \iff 0 = v(1) = v(x)+v(x^{-1}) \leq v(y)+v(x^{-1}) = v(yx^{-1}) \iff yx^{-1} \in A\backslash\{0\} \iff w(x) \leq w(y).$$

Hence, w is order preserving valuation. Moreover, we have an isomorphism by sending $w(x)$ to $v(x)$. To see it is isomorphism, notes that it sends $0, 1$ to $0, 1$, and it is bijective in a clear manner. Also, it is additively homomorphic, since $w(x) + w(y) = w(xy) \mapsto v(xy) = v(x) + v(y)$ which is sum of image of $w(x)$ and $w(y)$. Hence, it is bijective homomorphism thus isomorphism. Thus, v and w are essentially equivalent.

32. Since value group is defined as surjective image of K^*, so we can assume that Γ is image of K^* under a valuation map v.

Notes that $v(A - \mathfrak{p})$ is a monoid; to see this, $1 \in A - \mathfrak{p}$, thus $0 \in v(A - \mathfrak{p})$. And if $x, y \in A - \mathfrak{p}$, then $xy \in A - \mathfrak{p}$ since $\mathfrak{p}$ is prime, thus $v(x) + v(y) = v(xy) \in v(A - \mathfrak{p})$. Thus, $v(A - \mathfrak{p})$ is closed under addition. Now let $\Delta = -v(A - \mathfrak{p}) \cup v(A - \mathfrak{p})$. It suffices to show that Δ is isolated subgroup. Notes that any elements has inverse by construction. Thus, let $v(x), v(y) \in \Delta$. If both are positive or negative, then since $-v(A - \mathfrak{p})$ and $v(A - \mathfrak{p})$ are closed under addition, so their sum is in Δ. Now if $v(x) \in v(A - \mathfrak{p}), v(y) \in -v(A - \mathfrak{p})$, then $v(y) = -v(y')$ for some $y' \in A - \mathfrak{p}$. Hence, $y \in K^*$ is inverse of y' in the field of fraction K. Now, $v(x)+v(y) = v(xy)$. If $v(xy) \geq 0$, then $xy \in A$, hence $x/y' \in A$. This implies $x = x/y' \cdot y'$. This implies $x/y' \in A - \mathfrak{p}$. Thus, $xy \in A - \mathfrak{p}$, hence $v(x) + v(y) \in \Delta$. Otherwise, if $v(xy) < 0$, then $xy \notin A$, thus $x/y' \notin A$. Hence $y'/x \in A$, thus, $y'/x \cdot x = y'$ implies $y'/x \in A - \mathfrak{p}$, since both x, y' are not in $\mathfrak{p}$. Therefore, $y'/x = 1/xy \in A - \mathfrak{p}$. This implies $(xy)^{-1} \in A - \mathfrak{p}$, hence $v(xy) = -v(1/xy) \in \Delta$. Thus, in any case, Δ is closed under addition.

Thus Δ is a subgroup. Now we need to show that Δ is isolated. Let $x \in A - \mathfrak{p}$, and $y \in A$ such that $v(x) \geq v(y) \geq 0$. Then, $v(xy^{-1}) = v(x) - v(y) \geq 0$. This implies $xy^{-1} \in A$. If $y^{-1} \in A$, then y is unit, thus $y \in A - \mathfrak{p}$, done. Otherwise, x is nonunit. (If x is unit, then $y^{-1} \in A$, contradiction.) Also, $xy^{-1} \in A - \mathfrak{p}$, otherwise $xy^{-1} \cdot y = x \in \mathfrak{p}$, contradiction. Hence, $v(xy^{-1}) \in \Delta$. This implies $v(x) - v(y) \in \Delta$. Thus, $-v(y) \in \Delta$ since Δ is a subgroup containing $v(x)$ and $-v(x)$. Thus, $v(y) \in \Delta$ since Δ is closed under inversion. This shows Δ is isolated.

Now define a map Δ sending prime ideal $\mathfrak{p}$ to its corresponding isolated subgroup $\Delta(\mathfrak{p})$. If $\Delta(\mathfrak{p}) = \Delta(\mathfrak{q})$, then its subset whose sign is positive is also the same; this implies $v(A - \mathfrak{p}) = v(A - \mathfrak{q})$. Thus, $\forall x \in A - \mathfrak{p}, \exists y \in A - \mathfrak{q}$ such that $v(x) = v(y)$. Then, $0 = v(x) - v(y) = v(xy^{-1})$ implies $xy^{-1} \in A^*$ by construction, thus $x = xy^{-1} \cdot y \in A - \mathfrak{q}$ since unit is contained in $A \setminus \mathfrak{q}$ and from prime property of $\mathfrak{q}$. Thus, $A - \mathfrak{p} \subseteq A - \mathfrak{q}$. By the same argument in the other direction we can conclude that $A - \mathfrak{p} = A - \mathfrak{q}$, Thus, $\mathfrak{p} = \mathfrak{q}$. Hence, this mapping $\Delta(-)$ is injective.

To show that the mapping is bijective if we regard it as a map from $\mathrm{Spec}(A)$ to a set of all isolated subgroups of Γ, we need to show that it is surjective map. Let Δ be an isolated subgroup of Γ. Then, let $\mathfrak{p} = A \setminus v^{-1}(\Delta)$. If we show that $\mathfrak{p}$ is prime ideal, then $v(A - \mathfrak{p}) = v(A \cap v^{-1}(\Delta))$ is just a subset of Δ containing all nonnegative elements, thus Δ is the smallest subgroup containing it, hence $\Delta(\mathfrak{p}) = \Delta$, done. To see $\mathfrak{p}$ is prime ideal, let $x \in \mathfrak{p}$. Then, $-x \in \mathfrak{p}$ since $v(-x) = v(x)$. Also, if $x, y \in \mathfrak{p}$ then $v(x + y) = \min(v(x), v(y)) > v(z) \in \Delta$ for any $v(z) \in \Delta$ since $v(x), v(y) \notin \Delta$ and isolated property of Δ. Thus, $x + y \in \mathfrak{p}$. Thus $\mathfrak{p}$ is additive subgroup. Now, for any $x \in \mathfrak{p}$ and $y \in A$, $v(xy) = v(x) + v(y) \geq v(x) > v(z) \in \Delta$ for any $v(z) \in \Delta$, thus $xy \in \mathfrak{p}$. Hence $\mathfrak{p}$ is an ideal. Now, if $x, y \notin \mathfrak{p}$, then $x, y \in A \cap v^{-1}(\Delta)$, thus $v(xy) = v(x) + v(y) \in \Delta$, which implies $xy \notin \mathfrak{p}$. Thus $\mathfrak{p}$ is prime ideal.

Now to figure out what are the value groups of the valuation ring $A/\mathfrak{p}, A_{\mathfrak{p}}$, we need to define the value group. Let $v : A/\mathfrak{p} \to \Gamma$ by sending $\overline{a}$ to $v(a)$ if $\overline{a} \neq 0$, otherwise $\overline{p} = 0 \mapsto v(0)$ for any $p \in \mathfrak{p}$. To see it is well-defined, notes that if $\overline{a} = \overline{b}$, then $b - a = p$ for some $p \in \mathfrak{p}$, then $v(a) = v(b - p) \geq \min(v(b), v(-p)) = \min(v(b), v(0)) = v(b)$ since $v(0) = \infty$ by construction above. By the same argument, we get $v(b) \geq v(a)$, thus $v(\overline{a}) = v(\overline{b})$. Now let K' be a field of fraction of $A/\mathfrak{p}$. Then, extend v' such that sending $\overline{a}/\overline{b}$ to $v(a) - v(b)$. Then, still two properties of valuation map holds. (For multiplication,

$$v(a/b \cdot c/d) = v(ac) - v(bd) = v(a) + v(c) - v(b) - v(d) = v(a/b) + v(c/d)$$

and for addition,

$$v(a/b + c/d) = v((ad + bc)/bd) = v(ad + bc) - v(b) - v(d) \geq \min(v(ad), v(bc)) - v(b) - v(d).$$

If $v(ad)$ is minimum, then the righthandside is $v(a) - v(b) = v(a/b)$, otherwise it is $v(c/d)$, thus it is equal to $\min(v(a/b), v(c/d))$. Hence v is a valuation of K' by Exercise 5.31. Let $v(K') = \Delta$, which is a subgroup of Γ. Then, construct valuation ring using Exercise 5.31., say B. By construction, $B \cap A/\mathfrak{p} = \{\alpha \in B : v(\alpha) \geq 0\}$. Now we want to show that $B = A/\mathfrak{p}$.

Let $\alpha \in B$ such that $v(\alpha) \geq 0$. Then, $v(\alpha) = v(\overline{a}/\overline{b}) = v(\overline{a}) - v(\overline{b}) = v(a) - v(b) \geq 0$ for some $\overline{a}, \overline{b} \in A$ with nonzero nonunit $\overline{b}$. This implies that $ab^{-1} \in A$. If $ab^{-1} \in \mathfrak{p}$, then $a = ab^{-1} \cdot b \in \mathfrak{p}$, hence $\alpha = 0$. Otherwise, $ab^{-1} \in A - \mathfrak{p}$ implies $a = ab^{-1}b \in A - \mathfrak{p}$ by prime property of $\mathfrak{p}$. Hence we have $a, b, ab^{-1} \in A - \mathfrak{p}$. Thus, $v(\overline{ab^{-1}}) = v(\overline{a}/\overline{b})$. Now notes that in K', $\overline{ab^{-1}} = \overline{a}/\overline{b}$, since it has the same inverse $\overline{b}/\overline{a}$, i.e.,

$$\overline{ab^{-1}} \cdot \overline{b}/\overline{a} = \overline{a}/\overline{a} = \overline{1}.$$

Thus, $\alpha \in A/\mathfrak{p}$. Thus, $B = A/\mathfrak{p}$. Hence, $v(K')$ is the value group of $A/\mathfrak{p}$.

Now we claim that $v((K')^*) = \Delta(\mathfrak{p})$. First of all, $v(A/\mathfrak{p} \setminus \{0\}) \subseteq \Delta(\mathfrak{p})^+$ a subset of $\Delta(\mathfrak{p})$ having all nonnegative elements, by construction of extension. And for any nonnegative element in $\Delta(\mathfrak{p})$ comes from some $x \in A - \mathfrak{p}$, then $\overline{x} \neq 0$ in $A/\mathfrak{p}$ implies that $v(\overline{x})$ is the element. Hence, $v(A/\mathfrak{p}) \supseteq \Delta(\mathfrak{p})^+$, This shows that $v(A/\mathfrak{p} \setminus \{0\}) = \Delta(\mathfrak{p})^+$. SInce $\Delta(\mathfrak{p})$ is the smallest subgroup containing $\Delta(\mathfrak{p})^+$, $v((K')^*) \supseteq \Delta(\mathfrak{p})$. Conversely, any elements in $v((K')^*)$ is of form $v(a) - v(b) \in \Delta(\mathfrak{p})$, $v((K')^*) \subseteq \Delta(\mathfrak{p})$.

Also, in case of $A_\mathfrak{p}$, its field of fraction is the same as that of A. Hence, we can use the original valuation v on this case. Now, Proposition 5.18 says that A is local. Thus, $\mathfrak{m}$ contains $\mathfrak{p}$ and $U = A - \mathfrak{m}$, i.e., set of all units in A, by Proposition 1.6. And notes that $\Gamma = K^*/U$ by Exercise 5.30 and 5.31. Now notes that unit of $A_\mathfrak{p}$ is $V := \{a/b \in A_\mathfrak{p} : a, b \in A - \mathfrak{p}\}$. Hence, V is a subgroup of K^*. Thus, by Exercise 5.30, the value group of $A_\mathfrak{p}$ is K^*/V. Now notes that $V \subseteq U$ and both U, V are subgroup of K^*. Thus, by the third isomorphism theorem,

$$K^*/V \cong (K^*/U)/(V/U) \cong \Gamma/v(V)$$

since v is a canonical homomorphism $K^* \to K^*/U$.

First of all, we claim that $v(V) = v(A_\mathfrak{p} \setminus \{0\})$. To see this, notes that if $a/b \in A_\mathfrak{p}$ with $a \in \mathfrak{p}$, then $v(a/b) = v(a) - v(b)$

Now we claim that $v(V) = \Delta(\mathfrak{p})$. To see this, let $a/b \in V$. Then, $a, b \in A - \mathfrak{p}$. Thus $v(a/b) = v(a) - v(b) \in \Delta(\mathfrak{p})$ by construction of $\Delta(\mathfrak{p})$. Conversely, by construction, any element in $\Delta(\mathfrak{p})$ is of a form $v(a)$ or $-v(a)$ for some $a \in A - \mathfrak{p}$. Since $a/1, 1/a \in V$ $v(V)$ contains $v(a)$ and $-v(a)$, which implies $v(V) \supseteq \Delta(\mathfrak{p})$. Thus, the value group of $A_\mathfrak{p}$ is isomorphic to $\Gamma/v(V) = \Gamma/\Delta(\mathfrak{p})$.

33. To see A is an integral domain, notes that using the order of Γ, we can give a degree on A. Then, let $\phi = \sum_{g\in\Gamma} a_g x_g, \varphi = \sum_{g\in\Gamma} b_g x_g$ for some $a_g, b_g \in k$ all but finitely many zeros. Now suppose that x_{g_ϕ}, x_{g_φ} is the lowest nonzero term in ϕ and φ respectively. Then, $\phi\varphi$ has a lowest nonzero degree term $x_{g_\phi + g_\varphi}$, since $a_{g_\phi}, b_{g_\varphi}$ are nonzero, thus their product is nonzero since k is an integral domain. Hence, $\phi\varphi$ is nonzero.

   Actually, this is just proof of the claim that

   Claim XLV. *A group ring over integral domain is integral domain.*

   Now let $u = \lambda_1 x_{a_1} + \cdots + \lambda_n x_{a_n}$ is any non-zero element of A, where $\lambda_i \neq 0$ for all i, and $a_1 < \cdots a_n$. Define $v_0(u) = a_1$. Then, for any $u = \lambda_1 x_{a_1} + \cdots + \lambda_n x_{a_n}$, $v = \lambda'_1 x_{b_1} + \cdots + \lambda'_m x_{b_m}$, with $b_1 < \cdots < b_m$, uv has the lowest term $a_1 b_1 x_{a_1+b_1}$, thus $v_0(uv) = a_1 + b_1 = v_0(u) + v_0(v)$. Also, $v_0(u+v) \geq \min(a_1, b_1)$ where strict inequality occur when $a_1 = b_1$ and $\lambda_1 = -\lambda'_1$. Hence $v_0 : A - \{0\} \to \Gamma$ satisfies conditions of Exercise 5.31.

   Let K be the field of fraction of A. We need to show that v_0 uniquely extended to a valuation v of K, and that the value group of v is Γ.

   First of all, if v is an extension of v_0, then from axiom 1, $0 = v(1) = v(u/u) = v(u) + v(1/u)$. Thus, any extension v of v_0 has a value $v(1/u) = -v(u)$. Thus, any extension v should have $v(u/u') = v(u) - v(u')$, thus letting $v(a/b) := v_0(a) - v_0(b)$ is the unique extension if it is well-defined. And definitely, it is well-defined; if $a/b = c/d$,then $ad = bc$, thus $v_0(a) + v_0(d) = v_0(b) + v_0(c) \implies v_0(a) - v_0(b) = v_0(c) - v_0(d) \implies v(a/b) = v(c/d)$. So it is well-defined extension.

   To see that v is also a valuation, let $a/b, c/d \in K^*$. Then, $v(a/b \cdot c/d) = v_0(ac) - v_0(bd) = v_0(a) - v_0(b) + v_0(c) - v_0(d) = v(a/b) + v(c/d)$. Also, $v(a/b + c/d) = v((ad+bc)/bd) = v_0(ad+bc) - v_0(bd) \geq \min(v_0(ad), v_0(bc)) - v_0(bd)$, and if $v_0(ad)$ is minimum, then $\min(v_0(ad), v_0(bc)) - v_0(bd) = v_0(a) - v_0(b) = v(a/b)$, otherwise $v(c/d)$. Thus, $v(a/b + c/d) \geq \min(v(a/b), v(c/d))$. Thus v is a valuation.

   Now $v(K^*)$ is the value group of v by Exercise 5.30. We claim that $v(K^*) \cong \Gamma$. Notes that $v(A - \{0\}) = \Gamma$ since A contains x_g for any $g \in \Gamma$. Thus, $v(K^*) = \Gamma$ since K^* contains $A - \{0\}$. Hence, the value group of v is Γ.

   By Exercise 5.31, $B := \{u/u' \in K^* : v(u/u') \geq 0\}$ is the valuation ring of v. We claim that $B \neq A$ in general. Notes that $x_a \cdot x_{-a} = x_{a-a} = x_0$. This implies that $v(x_a) = -v(x_{-a})$. Thus, if a is nonzero elements in Γ, then either $v(x_a) > 0, v(x_{-a}) < 0$ or $v(x_a) < 0, v(x_{-a}) > 0$. In any case, if Γ has an element at least 3, i.e., an element whose order is not 2, then either $x_{-a} \notin B$ or $x_a \notin B$. Thus, $B \neq A$.

34. Let $C = g(B) \subseteq K$. Then, $C \supseteq A$ since $g(f(A)) = A$ in K. Also ,we can let $h : A \to C$ by $h = g \circ f$ is just canonical injection. By Proposition 5.18 i) and ii), C and A are local rings. Let $\mathfrak{n}$ be the maximal ideal of C. Notes that f is closed mapping, and g is homeomorphism onto $V(\ker(g))$, thus, $g^*(\mathfrak{n})$ is also maximal ideal, therefore

$$f^*(g^*(\mathfrak{n})) \underbrace{=}_{\text{Exercise 1.21 vi)}} (g \circ f)^*(\mathfrak{n}) = h^*(\mathfrak{n})$$

is maximal ideal since f preserves closed singleton to close singleton. Now, notes that $h^*(\mathfrak{n}) = h^{-1}(\mathfrak{n}) = \mathfrak{n} \cap A$ since h is an injection. Thus, C dominates A in the sense of Exercise 5.27, since $\mathfrak{n}$ is the unique maximal ideal of C. However, since $(C, \mathfrak{n}) \in \Sigma$ such that $(A, \mathfrak{m}) \leq (C, \mathfrak{n})$ but A is a valuation ring, $C = A$, by Exercise 5.27.

35. By exercise 3, $f : A \to B$ is integral implies $f \otimes 1 : A \bigotimes_A C \to B \bigotimes_A C$ is integral, thus by Exercise 1, $(f \otimes 1)^* \operatorname{Spec}(B \bigotimes_A C) \to \operatorname{Spec}(C)$ is closed map.

    Conversely, suppose that $f : A \to B$ is a ring homomorphism such that for any A-algebra C, $(f \otimes 1)^* : \operatorname{Spec}(B \bigotimes_A C) \to \operatorname{Spec}(C)$ is integral. Our goal is to prove that when B is integral domain, then f is integral. To follow the hints, let $A' = f(A)$. Then $f : A' \to B$ is just a canonical injection. Let K be the field of fractions of B. To use Corollary 5.22, it suffices to show that for any valuation ring A'' of K containing A contains B also. Since A'' is also an A-algebra by canonical map $A \to K$, we know $\operatorname{Spec}(A'' \bigotimes_A B) \to \operatorname{Spec}(A'' \bigotimes_A A) = \operatorname{Spec}(A'')$ is closed map. Now let $g : A'' \bigotimes_A B \to K$ by $a'' \otimes b \mapsto a''b$. It is well-defined A-algebra homomorphism, since it is induced by the universal property of tensor product applying on the bilinear map $A'' \times B \to K$ by $(a'', b) \mapsto a''b$. Then, by Exercise 5.34, $g(A'' \bigotimes_A B) = A''$. Then, $ba'' \in A''$ for all $b \in B$ and all $a'' \in A''$. Thus, by taking $a'' = 1$, $B \subseteq A''$. Since A'' was arbitrarily chosen valuation ring containing A, Corollary 5.22 shows that B is in the integral closure of A'. Thus, B is integral over A'. It is definition of the integral ring homomorphism.

    Now we change the condition, from B is integral domain to B is a ring with only finitely many minimal prime ideals. To see this, let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be all minimal prime ideals of B. Then, notes that $B \to B/\mathfrak{p}_i$ is surjective, which implies $B \bigotimes_A C \to B/\mathfrak{p}_i \bigotimes_A C$ is surjective for any A-algebra C, by Proposition 2.18 applying on the exact sequence $\ker(B \to B/\mathfrak{p}_i) \to B \to B/\mathfrak{p}_i \to 0$. Thus, $\operatorname{Spec}(B/\mathfrak{p} \bigotimes_A C) \to \operatorname{Spec}(B \bigotimes_A C)$ is closed map since $\operatorname{Spec}(B/\mathfrak{p}_i \bigotimes_A C) \cong V(\mathfrak{p}_i) \subseteq \operatorname{Spec}(B \bigotimes_A C)$, and any closed set of $V(\mathfrak{p}_i)$ is intersection of $V(\mathfrak{p}_i)$ with some closed set of $\operatorname{Spec}(B \bigotimes_A C)$, thus closed in $\operatorname{Spec}(B \bigotimes_A C)$. Hence, $A \to B \to B/\mathfrak{p}_i$ induces closed map $\operatorname{Spec}(B/\mathfrak{p}_i \bigotimes_A C) \to \operatorname{Spec}(C)$ for any A-algebra C. Thus, by apply the above results, we get $A \to B \to B/\mathfrak{p}_i$ is integral. Thus, $A \to B \to \prod_{i=1}^n (B/\mathfrak{p}_i)$ is integral by Exercise 5.6. Then kernel of $B \to \prod_{i=1}^n (B/\mathfrak{p}_i)$ is nilradical $\mathfrak{R}$ of B since intersection of minimal prime ideal is just intersection of all prime ideal. Thus, by the first isomorphism, $\prod_{i=1}^n (B/\mathfrak{p}_i) \cong B/\mathfrak{R}$ is integral. Now let $y \in B$. Then, there exists a monic polynomial $p(x) \in B/\mathfrak{R}[x]$ such that $p(\overline{y}) = 0$. Then, by picking any representatives of each coefficients of $p(x)$, we can regard $p(x)$ is in $f(A)[x]$ such that $p(y) \in \mathfrak{R}$. Then $\exists m \in \mathbb{N}$ such that $p(y)^n = 0$. Since $p(x)$ is monic, so is $p(x)^l$, and $p(y)^l = 0$ implies y is integral over $f(A)$. Thus, $f : A \to B$ is integral.

6 Chain Conditions

In the example in [3][p.75], to see that any subgroup of G are the G_n, notes that G_n is cyclic group generated by $1/p^n$. Now we use induction. Let H be a subgroup of G generated by one element, then done. If it is generated by n elements, then it has at least two generator a/p^n and b/p^m with irreducible fraction form, thus if we assume $n < m$ wlog, then by Bezout's identity, there is c, d such that $cap^{m-n} + db = 1$ since ap^{m-n} and b still are relatively prime. Thus, $d(b/p^m) + c(ap^{n-m}/p^m) = 1/p^m$, and definitely $\langle a/p^n, b/p^m \rangle \supseteq \langle 1/p^m \rangle$ and vice versa. Thus, we can reduce its generators. Thus, H is cyclic, done. If H has no finite set of generators, i.e., H is generated by infinitely many generators, then for any $m \in \mathbb{N}$, there exists $n > m$ such that a/p^n is in the generator of H, i.e., $gcd(a, p^n) = 1$. Since each a/p^n generates G_n, H contains G itself.

Claim XLVI. *Quotient of finitely generated module is finitely generated. Also, submoudles of quotient module bijectively correspond to submodules of original module containing the denominator.*

Proof. For any $m = \sum_{j=1}^{n} a_i g_i$ where $\{g_i\}$ is a set of generators of a finitely generated module M, $\phi(m) = \sum_{j=1}^{n} a_i \pi(g_i)$.

For M and M/N, every submodule of M containing N can be quotiented by N Conversely, for given submodule Q of M/N, $ker(M \to M/N \to (M/N)/Q)$ is a module corresponding to Q. ☐

Also, in a Proposition 6.9, additive is in a sense in the Category of A-module, i.e., $M = M' \oplus M''$, which gives a short exact sequence. And in Corollary 6.11, $\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_i$ is $A/\mathfrak{m}_i$ module by usual product; notes that if $\overline{x} \in \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_i$ and $\overline{a} = \overline{b} \in A/\mathfrak{m}$, then $a = b - m$ for some $m \in \mathfrak{m}_i$, thus $a\overline{x} = \overline{ax} = \overline{bx - mx} = \overline{bx} - \overline{mx} = \overline{bx}$ since $mx \in \mathfrak{m}_1 \cdots \mathfrak{m}_i$. Thus it is a vector space.

1. (a) Let $\mathfrak{a}_n = \ker(u^n)$ for each $n \in \mathbb{N}$. Then, $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ is a chain of ideals. Since M is Noetherian, $\exists n \in \mathbb{N}$ such that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$. Let $y \in \ker(u^{n+1}) = \ker(u^{n+1}) \cap M = \ker(u^{n+1}) \cap \operatorname{Im}(u^n)$. Then, $u^n(x) = y$ for some $x \in M$, and $u^{n+1}(x) = u(y) = 0$ implies $x \in \mathfrak{a}_{n+1} = \mathfrak{a}_n$, therefore $y = 0$. Hence, $\ker(u^{n+1}) = 0$, which implies that u^{n+1} is injective. Thus, u is injective on $\operatorname{Im}(u^n) = M$.

   (b) Let $\mathfrak{a}_n = \operatorname{coker}(u^n)$ for each $n \in \mathbb{N}$. Then, $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ is a chain of ideals. Since M is Artinian, $\exists n \in \mathbb{N}$ such that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$. This implies $\operatorname{Im}(u^n) = \operatorname{Im}(u^{n+1}) = \cdots$. Thus, for any $x \in M$, $\exists y \in M$ such that $u^n(x) = u^{n+1}(y)$, thus $u^n(x - u(y)) = 0$. Since u is injective, so is u^n, thus $x = u(y)$. Since x was chosen arbitrary, u is surjective.

2. Suppose M is not Noetherian. By Proposition 6.2, M has a submodule N which is not finitely generated. Pick $x_1 \in N$, then $N - Ax_1 \neq \emptyset$, thus pick $x_2 \in N - Ax_1$. Since $N \neq (Ax_1 + Ax_2)$, pick $x_3 \in N - (Ax_1 + Ax_2)$, and so on. Then, $\{Ax_1, Ax_1 + Ax_2, \cdots\}$ is a countable set which doesn't contain a maximal element, contradiction. (Notes that we can construct a maximal one, which is union of all modules in the set; however, this union itself is not contained in a set, since it is not finitely generated.) Thus M is Noetherian.

3. Proposition 2.1 ii) implies that $(N_1 + N_2)/N_1 \cong N_2/(N_1 \cap N_2)$. Since $(N_1 + N_2)/N_1$ is a submodule of M/N_1, it is Noetherian, thus $N_2/(N_1 \cap N_2)$ is Noetherian. Now take an exact sequence
$$0 \to N_2/(N_1 \cap N_2) \to M/(N_1 \cap N_2) \to M/N_2 \to 0.$$
Since $M/N_2 \cong (M/(N_1 \cap N_2))/N_2/(N_1 \cap N_2)$, it is exact. Also, since both $N_2/(N_1 \cap N_2)$ and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$ by Proposition 6.3.

   Conversely, suppose $M/N_1, M/N_2$ are Artinian. Then by the same argument, just replacing the word 'Noetherian' to 'Artinian' we get the desired result, i.e., $M/(N_1 \cap N_2)$ is Artinian.

4. Since M is Noetherian, it is finitely generated by Proposition 6.2, thus there exists $m_1, \cdots, m_n$, generators of M. Then, $\mathfrak{a} = \bigcap_{i=1}^{n} \operatorname{Ann}(m_i)$. Since $A/\operatorname{Ann}(m_i) = Ax_i \subset M$, $A/\operatorname{Ann}(m_i)$ is Notherian for any i. Thus, $A/\mathfrak{a}$ is Noetherian when $n = 2$ by Exercise 6.3. Now suppose it holds for $2, 3, \cdots, k-1$. Then, $A/\bigcap_{i=1}^{k} \operatorname{Ann}(m_i) = A/(\bigcap_{i=1}^{k-1} \operatorname{Ann}(m_i) + \operatorname{Ann}(m_i))$ and $A/(\bigcap_{i=1}^{k-1} \operatorname{Ann}(m_i))$ is Noetherian by inductive hypothesis. Thus, by applying Exercise 6.3 again, we get $A/\bigcap_{i=1}^{k} \operatorname{Ann}(m_i)$ is Noetherian. Thus by induction, $A/\mathfrak{a}$ is Noetherian.

   However, this doesn't hold when we replace Noetherian by Artinian. Let G be a group G in Example 3) of [3][p.75] Then, G is Artinian $\mathbb{Z}$-module, and $\operatorname{Ann}_{\mathbb{Z}}(G_n) := p^n$ since $G_n := \langle \overline{1/p^n} \rangle$. However, $\mathfrak{a} = \operatorname{Ann}(G) = \bigcap_{i=1}^{\infty} \operatorname{Ann}_{\mathbb{Z}}(G_i) = 0$, thus $\mathbb{Z}/\mathfrak{a} = \mathbb{Z}$, but $\mathbb{Z}$ is not artinian, since it has infinte length chain $(p) \supseteq (p^2) \supseteq \cdots$.

5. Let $U_1 \subseteq U_2 \subseteq \cdots$ be a chain of open sets in a subspace X_0. Then, $U_i = V_i \cap X_0$ for some open set V_i for each i, hence it induces a chain of open sets $V_1 \subseteq V_2 \subseteq \cdots$ in X, thus it stabilizes, therefore $U_1 \subseteq U_2 \subseteq \cdots$ stabilizes.

   To see that X is quasi compact, suppose that $\{U_\alpha\}_{\alpha\in I}$ be a open cover of X. Pick an open set and define it as U_1. Define U_n be union of U_{n-1} and an open set V chosen from $\{U_\alpha\}_{\alpha\in I}$ such that

   $$V = \begin{cases} \text{an open set in the cover } \{U_\alpha\}_{\alpha\in I} \text{ such that } V \setminus U_{n-1} \neq \emptyset \\ \emptyset \text{ if a set satisfying the above condition doesn't exist.} \end{cases}$$

   Then, U_n is still open since it is finite union of open sets, thus $U_1 \subseteq U_2 \subseteq \cdots$ is an ascending chain of open sets in X. Since X is Noetherian, it stabilizes at some $n \in \mathbb{N}$. Now suppose $X \neq U_n$ Then, $\exists x \in X \setminus U_n$. Hence, we have an open neighborhood $V' \in \{U_\alpha\}_{\alpha\in I}$ of x. Thus $U_{n+1} \supsetneq U_n$, contradiction. Hence, $U_n = X$. Therefore, $\{U_\alpha\}_{\alpha\in I}$ has finite subcover of X.

6. $i) \implies iii)$: By Exercise 6.5, every subspace of X is Noetherian, thus quasi-compact.

   $iii) \implies ii)$: Clear.

   $ii) \implies i)$: Since X is also an open subspace of X, done. (Or, if we take any ascending chain of open sets, let U be union of all ideals in a chain. Then, U is open in X since for any $x \in U$, it is contained in an open set in the chain, thus at least one of its open neighborhood is inside of U. By ii) U is quasi-compact. Since the given ascending chain is a covering of U, it has finite subcover, such that $U_{i_1}, \cdots, U_{i_m}$ with $i_1 < \cdots < i_m$. Thus, by letting $n = i_m$, we can see that $U_n = \bigcup_{i=1}^n U_i = U \supseteq U_{n+k} \supseteq U_n$ for any k. This implies $U_{n+k} = U_n$ for any k.

7. Suppose not. Then, there is a Noetherian space X such that X is not a finite union of irreducible closed subspaces. Thus, Σ defined in the hint of this Exercise is nonempty. By d.c.c. of Noetherian space and Proposition 6.1, Σ has a minimal element, say X_0. Then, X_0 is reducible, otherwise it is trivially a finite union of irreducible close subspace. Thus, by Claim II, $X_0 = C_1 \cup C_2$ for some proper closed set of X. However, since X_0 is a minimal closed subspace which is not a finite union of irreducible closed subspaces, each C_1 and C_2 is a finite union of irreducible closed subspaces, so is X_0, contradiction. Thus every Noetherian space is a finite union of irreducible closed subspaces.

   For the last statement, by Exercise 1.20 iii), there exist a set of all maximal irreducible subspaces, i.e., set of all irreducible components of X, $\mathscr{Y} := \{Y_i\}_{i\in I}$ of X such that $\mathscr{Y}$ is closed covering of X and each Y_i is closed. Let $X = \bigcup_{i=1}^n C_i$ for some irreducible closed subspaces of X for some $n \in \mathbb{N}$. Let Y_i be a maximal irreducible subspaces containing C_i. (Such Y_i exists by Exercise 1.20 ii).) Then, $X = \bigcup_{i=1}^n Y_i$. Hence $\{Y_i\}_{i=1}^n$ is a finite subcover of $\mathscr{Y}$. Now suppose that n be the minimal number of the cardinality of finite subcovering of $\mathscr{Y}$. Thus, for any maximal irreducible subspaces Y_j , $Y_j = \bigcup_{i=1}^n Y_j \cap Y_n$, thus $Y_j = Y_i$ for some $i = 1, \cdots, n$ by maximality. Thus, $\mathscr{Y}$ is finite.

8. It suffices to show that a collection of closed sets in $\mathrm{Spec}(A)$ satisfy d.c.c. Let $V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \cdots$ be a descending chain of closed sets in $\mathrm{Spec}(A)$. Then, $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq$ is a chain of ideals in A. Since A is Noetherian, it stabilizes at some $n \in \mathbb{N}$. Hence, $V(\mathfrak{a}_n) = V(\mathfrak{a}_{n+k})$ for any k. Thus d.c.c. holds.

   Converse is not true in general. Let $A = k[x_1, \cdots,]/(x_1^2, x_2^2, \cdots)$. Then, nilradical of A is $(x_1, x_2, \cdots)$, thus $A/(x_1, x_2, \cdots) \cong k$ implies $\mathrm{Spec}(A/\mathfrak{R})$ is a singleton, thus Noetherian. By Exercise 1.21 iv), $\mathrm{Spec}(A) \cong \mathrm{Spec}(A/\mathfrak{R})$ is Noetherian. However, A is not Noetherian since $(x_1) \subseteq (x_1, x_2) \subseteq \cdots$ is an infinite chain.

9. By Exercise 6.8, $\mathrm{Spec}(A)$ is Noetherian, thus by Exercise 6.7, $\mathrm{Spec}(A)$ is a finitely many irreducible components. In the proof of Exercise 1.20 iv), we shows that every irreducible closed subset of $\mathrm{Spec}(A)$ is of form $V(\mathfrak{p})$. Thus $V(\mathfrak{p})$ is irreducible components if and only if $\mathfrak{p}$ is minimal prime. Hence, A has only finitely many minimal prime ideal.

10. Notes that $\mathrm{Supp}(M)$ is a set of prime ideal $\mathfrak{p}$ such that $M_\mathfrak{p} \neq 0$. Let $\mathfrak{a} = \mathrm{Ann}(M)$. Then, by Exercise 3.19 v), $\mathrm{Supp}(M) = V(\mathfrak{a})$, thus it is closed. By Exercise 1.21 vi), $V(\mathfrak{a}) \cong \mathrm{Spec}(A/\mathfrak{a})$ as a homeomorphism. By Exercise 6.4, $A/\mathfrak{a}$ is Noetherian. Hence, $\mathrm{Supp}(M)$ is Noetherian subspace of $\mathrm{Spec}(A)$.

11. By Exercise 5.10, it suffices to show that going-up property implies closed mapping when Spec(B) is Noetherian. Suppose f has going up property and Spec(B) is Noetherian. Let $V(\mathfrak{b})$ be a closed set in Spec(B). Then, Exercise 6.5 implies that $V(\mathfrak{b})$ is Noetherian. By Exercise 1.21 vi), Spec($B/\mathfrak{b}$) is homeomorphic to $V(\mathfrak{b})$, thus Noetherian. Hence by Exercise 6.7 and Exercise 1.20 iv), $B/\mathfrak{b}$ has finitely many minimal primes. Thus its contraction, say $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ are minimal elements in $V(\mathfrak{b})$, thus $V(\mathfrak{b}) = \bigcup_{i=1}^n V(\mathfrak{q}_i)$. Now let $\mathfrak{p}_i = f^*(\mathfrak{q}_i)$. Then, $r(\mathfrak{b})^c = \bigcap_{i=1}^n \mathfrak{p}_i$ using Exercise 1.18. Thus any prime ideal $\mathfrak{p}$ in $V(r(\mathfrak{b})^c)$ contains $\bigcap_{i=1}^n \mathfrak{p}_i$, so Proposition 1.11 ii) says $\mathfrak{p} \supseteq \mathfrak{p}_j$ for some $j \in [n]$. Thus, $V(r(\mathfrak{b})^c) \subseteq \bigcup_{i=1}^n(\mathfrak{p}_i)$. And the other inclusion is clear since each $\mathfrak{p}_i$ contained in the lefthandside. Thus, $V(r(\mathfrak{b})^c) = \bigcup_{i=1}^n(\mathfrak{p}_i)$. By Exercise 5.10 i), going up property implies that $V(\mathfrak{p}_i) = f^*(V(\mathfrak{q}_i))$. Hence,
$$f^*(V(\mathfrak{b})) = f^*(\bigcup_{i=1}^n V(\mathfrak{q}_i)) = \bigcup_{i=1}^n (\mathfrak{p}_i) = V(r(\mathfrak{b})^c).$$
Thus, f^* preserves closedness.

12. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots$ be an ascending chain or prime ideals in A. This induces a descending chain of closed sets $V(\mathfrak{p}_1 \supseteq V(\mathfrak{p}_2) \supseteq \cdots$, thus it stabilizes, which implies the chain or prime ideal stabilizes.

    For the counterexample, I just refer [8]. Let A be the counterble direct product of $\mathbb{Z}/2\mathbb{Z}$. Then, Spec(A) is countable disjoint union of $X_i = \text{Spec}(\mathbb{Z}/2\mathbb{Z})$ for any $i \in \mathbb{N}$. Notes that the set of prime ideals of A is $\{(0) \times \prod_{i \neq j}^\infty \mathbb{Z}/2\mathbb{Z} : j \in \mathbb{N}\} \cup \{0\}$, thus every elements except 0 is maximal. Hence, every ascending chain stabilizes. However, Spec(A) has the infinite ascending chain of open sets, by unioning those disjoint unions. (For example, $X_1 \subseteq X_1 \cup X_2 \subseteq \cdots$.)

7 Noetherian Rings

In proposition 7.8, (i) implies (ii) by Proposition 5.1 and (ii) implies (i) using generators of C as a finitely generated A-module and showing that its monic polynomial over B induces a finitely many generators of C as a B-module.

Claim XLVII. *If B is a finitely generated A-algebra and C is a finitely generated B-module, then C is a finitely generated A-algebra.*

Proof. Let x_i, $i = 1, \cdots, n$ be generators of C as a B-module. Then, $c = \sum_{i=1}^n b_i x_i$ for some $b_i \in B$. And every element in B is just polynomial over y_i's $i = 1, \cdots, m$ with coefficient from A. Thus, $b_i = f(y_1, \cdots, y_m)$ for some n. Thus, every element in C is a polynomial over $y_1, \cdots, y_m, x_1, \cdots, x_n$ with coefficients from A. Thus, C is a finitely generated A-algebra (by letting $x_i x_j = 0$ for any i, j.) ☐

1. Since A is not Noetherian, Σ is nonempty. Now let $\{\mathfrak{a}_i\}$ be a chain of ideal in Σ. Let $\mathfrak{a} = \bigcup_i \mathfrak{a}_i$. Then $\mathfrak{a}$ is an ideal (We did this kind of argument in the previous problems.) Moreover, $\mathfrak{a}$ is not finitely generated; if it was, then $\langle x_1, \cdots, x_n \rangle = \mathfrak{a}$. Then, $\exists m \in \mathbb{N}$ such that $x_1, \cdots, x_n \in \mathfrak{a}_m$ by definition of $\mathfrak{a}$. However, this implies $\mathfrak{a} \subseteq \mathfrak{a}_m = \mathfrak{a}$, hence $\mathfrak{a}_m$ is finitely generated, contradiction. Thus, $\mathfrak{a} \in \Sigma$. Hence, by Zorn's lemma Σ has a maximal element.

    Now let $\mathfrak{p}$ be such a maximal element in Σ. Suppose it is not a prime. Then, $xy \in \mathfrak{p}$ but $x, y \notin \mathfrak{p}$ exists. Then, $\mathfrak{a} + (x)$ contains $\mathfrak{a}$ strictly, thus it is not in Σ (otherwise, $\mathfrak{a}$ is not maximal.) Thus, $\mathfrak{a} + (x)$ is finitely generated, thus $\mathfrak{a}_0 + (x) = \mathfrak{a} + (x)$ for some finitely generated ideal $\mathfrak{a}_0$. Also, any minimal generator of $\mathfrak{a}_0$ should be contained in $\mathfrak{a}$ since they are not in (x). This implies $\mathfrak{a}_0 \subseteq \mathfrak{a}$.

    Also, as hint suggested, $\mathfrak{a} \supseteq \mathfrak{a}_0 + x(a : x)$ and if $y \in \mathfrak{a} \subseteq \mathfrak{a} + (x) = \mathfrak{a}_0 + (x)$, then $y = y_0 + xz$ for some $z \in A, y_0 \in \mathfrak{a}_0$. This implies $xz \in \mathfrak{a}$, thus $z \in (\mathfrak{a} : x)$. Hence, $y \in \mathfrak{a}_0 + x(a : x)$, thus $\mathfrak{a} = \mathfrak{a}_0 + x(a : x)$. Since $y \in (\mathfrak{a} : x)$, $(\mathfrak{a} : x)$ strictly contains $\mathfrak{a}$, it is finitely generated. This implies $\mathfrak{a}$ is finitely generated, contradiction.

2. By Exercise 1.5 ii), if a_n is nilpotent for any n, then f is nilpotent. Conversely, by Corollay 7,15, $\exists m \in \mathbb{N}$ such that $\mathfrak{R}^m = (0)$. Hence, $f^m = 0$.

3. $i) \implies ii)$: By Proposition 4.8, either $(S^{-1}\mathfrak{a})^c = \mathfrak{a} = (\mathfrak{a} : 1)$ or $A = (\mathfrak{a} : a)$ for some $a \in A \cap S$. Pick $x = 1$ or a.

   $ii) \implies iii)$: Let $S := \{1, x, x^2, \cdots\}$ for fixed $x \in A$. Then, $(S^{-1}\mathfrak{a})^c = (\mathfrak{a} : x^k)$ for some $k \in \mathbb{N}$. Also, by definition,

   $$\bigcup_{i\in\mathbb{N}} (\mathfrak{a} : x^i) = \{a \in A : x^i a \in \mathfrak{a}\} = \{a \in A : a/1 = b/x^i \text{ for some } b \in \mathfrak{a}\} = \{a \in A : a/1 \in \mathfrak{a}^e\} = (S^{-1}\mathfrak{a})^c.$$

   Hence, $(\mathfrak{a} : x^k) = \bigcup_{i\in\mathbb{N}} (\mathfrak{a} : x^i)$, thus the sequence is stationary. Since x was chosen arbitrarily, done.

   $iii) \implies i)$: Without loss of generatliy, assume $\mathfrak{a} = (0)$ (by thinking $A/\mathfrak{a}$ as a ring instead of A.) Let $xy = 0 \in (0)$ and $y \neq 0$. Then, $\mathrm{Ann}(x^n)$ is stationary, this implies that $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1})$ for some $n \in \mathbb{N}$. Now, let $a \in (y) \cap (x^n)$. Then, $ax = 0$ and $a = bx^n$. Thus, $bx^{n+1} = 0$ implies $b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$, thus $a = bx^n = 0$. Thus, $(y) \cap (x^n) = 0$. Since (0) is irreducible and $(y) \neq 0$, $x^n = 0$. This shows that (0) is primary.

4. I refer [4] for this exercise.

   (a) Let A be the ring given in the problem. Then, any element in A is a form $p(z)/q(z)$ with $p(z), q(z) \in \mathbb{C}[z]$ such that $q(z)$ has no zero on $|z| = 1$. Thus, $q(z)$ is not in an ideal $\mathfrak{p} = (z-a)_{|a|=1}$. Notes that $\mathfrak{p}$ is prime; since if $fg \in \mathfrak{p}$ but $g \notin \mathfrak{p}$, then by splitting f and g into linear ones, we can conclude that g does not contain any of $(z - a)$ for any $|a| = 1$. Thus, f should be divisible by one of $(z - a)$, thus $f \in \mathfrak{p}$. Hence, $S = \mathbb{C}[z] - \mathfrak{p}$ is a multiplicatively closed set. Now, $A = S^{-1}\mathbb{C}[z]$ by construction. Since $\mathbb{C}$ is Noetherian, $\mathbb{C}[z]$ is Noetherian by Hilbert Basis theorem, thus its localization A is Noetherian by Proposition 7.3.

   (b) Let A be the ring given in the problem. Then,

   $$A := \{f \in \mathbb{C}[[z]] : f = \sum_{n\in\mathbb{N}} c_n x^n \text{ with } \limsup_{n\to\infty} \sqrt[n]{|c_n|} < \infty\} \subseteq \mathbb{C}[[z]].$$

   Notes that A is closed under addition and multiplication. (To see this, just pick x from intersection of convergence interval of two functions in A and plug in to the subtraction or product of them. This implies they have positive convergence of radius, thus are in A.)

   Now we claim that

   Claim XLVIII. *In $\mathbb{C}[[z]]$, $(f) = (z^{\mathrm{ord}(f)})$ where $\mathrm{ord}(f)$ is the smallest $n \in \mathbb{N}$ such that $c_n \neq 0$. Thus, any ideal in $\mathbb{C}[[z]]$ is (z^k) for some $k \in \mathbb{N} \cup \{0\}$ or 0. Moreover, if $f = z^{\mathrm{ord}(f)}g$ with g is unit, then g has the radius of convergence greater than or equal to f.*

   Proof. Notes that $f = z^{\mathrm{ord}(f)}g$ for some g with $\mathrm{ord}(g) = 0$. Thus, g has nonzero constant term. By Exercise 1.5 i), g is unit in $\mathbb{C}[[z]]$. Hence, $fg^{-1} = z^{\mathrm{ord}(f)}$. Hence, since $\mathbb{C}[[z]]$ is Noetherian, every ideal is finitely generated, then we can assume that those generators are comes from $\{0, 1, z, z^1, \cdots\}$, therefore it is principal.

   For the second one, notes that for any x which is in the interval of convergence of f, $x^{\mathrm{ord}(f)}$ is convergent. Thus, $g(x)$ should be convergent series. Since x was chosen arbitrarily, g has the radius of convergence greater than or equal to f. ☐

   Now, let $f \in A$. Then, $f = z^m g$ by the proof of above claim for some $m \in \mathbb{N}, g$ is unit with $g \in A$. Thus, $(f) = (z^m)$ in A. Hence A is PID, thus Noetherian by definition.

   (c) Let A be the ring given in the problem. Then,

   $$A := \{f \in \mathbb{C}[[z]] : f = \sum_{n\in\mathbb{N}} c_n x^n \text{ with } \limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0\} \subseteq \mathbb{C}[[z]].$$

   By the same argument, A is a subring of $\mathbb{C}[[z]]$.

   Notes that $\sin(z)/z = \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2\pi^2}$. Let $f_n := \prod_{j=n}^{\infty}(1 - \frac{z^2}{n^2}$. Then, $\pi z f_1 = \sin(\pi z)$. Thus, $\lim_{z\to n} |f_n| = 0$. Suppose $\exists g \in \mathbb{C}[[z]]$ such that $gf_n(n) \neq 0$. Then, $\lim_{z\to n} |g(z)| = \infty$. Thus

$g \notin A$ since it should have the radius of convergence less than n. Thus, from $f_{n+1}|f_n$, we know (f_n) is contained in (f_{n+1}). However, $f_n g = f_{n+1}$ implies $g \notin A$ by this argument. Thus, (f_n) is strictly contained in (f_{n+1}). This shows that A has infinite ascending chain, contradiction.

(d) In this case, the desired ring is $A = \mathbb{C} + (z^{k+1}) \subsetneq \mathbb{C}[z]$. To see this, let $f = \sum_{n\in\mathbb{N}} c_n z^n$ be such a polynomial. Since first k derivatives vanish at the origin implies that $c_1 = \cdots = c_k = 0$. Thus, $f \in \mathbb{C} + (z^{k+1})$. Other direction is trivial. Notes that $\mathbb{C}[z^{n+1}]$ is Noetherian by the Hilbert basis theorem (with variable z^{n+1}.) Then A is $\mathbb{C}[z^{n+1}]$ module generated by $\{1, z, \cdots, z^n\}$. Hence, A is finitely generated, therefore Noetherian by Proposition 7.2.

(e) In this case,

$$A = \{f(z,w) \in \mathbb{C}[z,w] : \frac{\partial}{\partial w^n}|_{z=0} f = 0 \text{ for any } n.\}.$$

A is a ring since product rule of derivation and addition rule of derivation gives that A is closed under subtraction and multiplication. Let $f \in A$. Then, $f = \sum_{n\in\mathbb{N}} f_n(z) w^n$ for some $f_n(z) \in \mathbb{C}[z]$. Let $a_n = f_n(0)$. Then,

$$\frac{\partial}{\partial w^n}|_{z=0} f = \sum_{j=n}^{m} a_j j^{\underline{n}} w^{j-n} = 0.$$

This implies $a_j = 0$ for all $j \geq n$. Since $n \in \mathbb{N}$, this implies $a_j = 0$ for all $j > 0$. Thus, $A = \mathbb{C}[z] + (z)\mathbb{C}[w]$. Thus, $w^n \notin A$ for any n. Moreover, $zw^n | zw^{n+1}$ implies $(zw, \cdots, zw^n)$ contained in $(zw, \cdots, zw^{n+1})$ strictly. Thus, A has an infinite ascending chain of ideals. Thus A is not Noetherian.

5. Exericse 5.12 shows that B is integral over B^G. This implies $A \subseteq B^G \subseteq B$. Also, A is Noetherian, B is finitely generated as an A-algebra, and B is integral over B^G. Thus we can apply Proposition 7.8 to conclude that B_0 is finitely generated as an A-algebra.

6. If K has characteristic 0, then $\mathbb{Z} \subset \mathbb{Q} \subseteq K$. (To see this, $\mathbb{Z} \to K$ by $1_\mathbb{Z} \to 1_K$ is injective, thus we can define $\mathbb{Q} \to K$ an injective field homomorphism, which gives a subfield of K isomorphic to $\mathbb{Q}$.) Since K is finitely generated over $\mathbb{Z}$ (every ring is $\mathbb{Z}$-algebra), K is finitely generated over Q. Thus, by Proposition 7.9, K is a finite algebraic extension of $\mathbb{Q}$. However, Proposition 7.8 shows that K is finitely generated as $\mathbb{Z}$-algebra. This implies that $\mathbb{Q}$ is finitely generated as $\mathbb{Z}$-algebra, contradiction.

   If K is characteristic $p \neq 0$, then K contains $\mathbb{Z}/p\mathbb{Z}$ as a subfield by the map $\mathbb{Z} \to K$ by $1_\mathbb{Z} \to 1_K$. Since $\mathbb{Z}$ acts on K through $\mathbb{Z}/p\mathbb{Z}$, K is also finitely generated $\mathbb{Z}/p\mathbb{Z}$-algebra. Then Proposition 7.9 implies that K is a finite extension of $\mathbb{Z}/p\mathbb{Z}$. Hence K is a finite field.

7. Corollary 7.6 implies that $k[t_1, \cdots, t_n]$ is Noetherian. Thus, an ideal generated by f_α with $\alpha \in I$ is finitely generated. Let $g_1, \cdots, g_n$ such generator of the ideal. Then, each g_i is a linear combination of f_α with coefficients from $k[t_1, \cdots, t_n]$. Thus, by collecting all f_α used in the linear combination of each g_i in a set A, we can see that an ideal generated by A contains the whole ideal, thus $A \subseteq \{f_\alpha : \alpha \in I\}$ generates A.

8. Since $A[x]/(x) \cong A$, Proposition 6.6 shows that A is Noetherian if $A[x]$ is Noetherian.

9. Let $\mathfrak{a} \neq 0$ be an ideal in A. Let $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ be the maximal ideals which contains $\mathfrak{a}$. Choose $x_0 \neq 0$ in $\mathfrak{a}$ and let $\mathfrak{m}_1, \cdots, \mathfrak{m}_{r+s}$ be the maximal ideals which contains x_0. Since $\mathfrak{m}_{r+1}, \cdots, \mathfrak{m}_{r+s}$ do not contain $\mathfrak{a}$, there exists $x_j \in \mathfrak{a}$ such that $x_j \notin \mathfrak{m}_{r+j}$ for $j \in [s]$. Since each $A_{\mathfrak{m}_i}$ is Noetherian for $i \in [r]$, the extension of $\mathfrak{a}$ in $A_{\mathfrak{m}_i}$ is finitely generated. Thus, there exists $x_{s+1}, \cdots, x_t$ in $\mathfrak{a}$ whose image in $A_{\mathfrak{m}_i}$ generates the extension of $\mathfrak{a}$ for all $i \in [r]$. Let $\mathfrak{a}_0 = (x_0, \cdots, x_t)$. Then, $\mathfrak{a}_0$ and $\mathfrak{a}$ have the same extension in $A_\mathfrak{m}$, since, if $\mathfrak{m} = \mathfrak{m}_i$ for $i \in [r]$, done. Otherwise, if $i = r+1, \cdots, s$, then $\mathfrak{a}_0^e = (1) = \mathfrak{a}^e$ since $\mathfrak{a}$ and $\mathfrak{a}_0$ an element outside of $\mathfrak{m}_i$. For any other maximal ideal $\mathfrak{m}$, each $\mathfrak{a}$ and $\mathfrak{a}_0$ also has an element in $A - \mathfrak{m}$, thus their extension is (1). Thus, extension of $\mathfrak{a}$ and $\mathfrak{a}_0$ are the same in the localization by any maximal ideal. By Proposition 3.9, this implies $\mathfrak{a}_0 = \mathfrak{a}$.

10. In Exercise 2.6, $M[x] \cong M \bigotimes_A A[x]$. Notes that M and $A[x]$ are Noetherian (by Hilbert basis theorem.) Thus, $M \oplus A[x]$ is Noetherian by Corollary 6.4. Since $M \bigotimes_A A[x]$ is homomorphic image of $M \oplus A[x]$, thus it gives an exact sequence

$$0 \to \ker(M \oplus A[x] \to M \bigotimes_A A[x]) \to M \oplus A[x] \to M \bigotimes_A A[x] \to 0.$$

By Proposition 6.3, this implies that $M \bigotimes_A A[x]$ is Noetherian.

11. Let $A = \prod_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. As we've seen in the proof of the Exercise 6.12, A is not Noetherian. Now let $\mathfrak{p}$ be an arbitrary prime ideal of A and think $A_\mathfrak{p}$. If $x \in A_\mathfrak{p}$, then x is of form $(x_0, \cdots)/(y_0, \cdots)$ with $y = (y_0, \cdots) \in A - \mathfrak{p}$. Also notes that $x^2 = x$ since $(x_0^2, \cdots) = (x_0, \cdots)$ for any elements $(x_0, \cdots)$ in A. Assume that $x \in \mathfrak{p}A_\mathfrak{p}$. Then, still, $x^2 - x = 0$. This implies $(x-1)x = 0$ where 1 denotes multiplicative identity of $A_\mathfrak{p}$. If x is nonzero, then $(x-1)$ must be zero since A is integral domain, thus localization of it is also integral domain. This implies that there exists $(z_0, \cdots) \in A - \mathfrak{p}$ such that

$$(z_0, \cdots)(x_0, \cdots) = (y_0, \cdots).$$

However, since $(x_0, \cdots)$ is in $\mathfrak{p}$, this implies $(y_0, \cdots) \in \mathfrak{p} \cap A - \mathfrak{p}$, contradiction. Thus, $\mathfrak{p}A_\mathfrak{p} = 0$. Thus, any localization of A by prime ideal is actually a field. Therefore $A_\mathfrak{p}$ is Noetherian.

12. By Exercise 3.16 condition i), $\mathfrak{a}^{ec} = \mathfrak{a}$ for all ideal $\mathfrak{a}$ of A. Thus, $\mathfrak{a} \mapsto \mathfrak{a}^e$ is an injective map. Hence let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ be an ascending chain of ideal in A. Then, $\mathfrak{a}_1^e \subseteq \mathfrak{a}_2^e \subseteq \cdots$ is an ascending chain of ideal in B, which must be stationary at some points. Thus, its contraction, which is the original ascending chain in A, must be stationary from injectivity of the map $\mathfrak{a} \mapsto \mathfrak{a}^e$. Hence A is Noetherian.

13. By Exercise 3.21 iv), fibers of f^* at $\mathfrak{p} \in \operatorname{Spec}(B)$ is $\operatorname{Spec}(k(\mathfrak{p}) \bigotimes_A B) = \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$. Also, since f is of finite type, B is finitely generated A-algebra, thus B is finitely generated $f(A)$-module. Since $f(A)$ is Noetherian by Proposition 7.1., B is also Noetherian by Proposition 7.2. Thus by Proposition 7.3, $B_\mathfrak{p}$ is Noetherian, therefore $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is Noetherian by Proposition 6.6. By Exercise 6.8, $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ is Noetherian space.

    (Or we can conclude that since $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is a field, thus Noetherian, therefore $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ is Noetherian space.)

14. If $f^n \in \mathfrak{a}$, then $f^n(V) = \{0\}$ by construction of V. Therefore, $f^n \in I(V)$, thus,$r(\mathfrak{a}) \subseteq I(V)$. Conversely, let $f \notin r(\mathfrak{a})$. Then $\exists \mathfrak{p}$, a prime ideal containing $\mathfrak{a}$ such that $f \notin \mathfrak{p}$ by claim XIV. Let $\overline{f}$ be the image of f in $B = A/\mathfrak{p}$. Let $C = B_f = B[1/\overline{f}]$. Let $\mathfrak{m}$ be a maximal ideal of C. Since A is finitely generated k-algebra, so does B, so does C and $C/\mathfrak{m}$. (Generators are image of $t_1, \cdots, t_n$ in $C/\mathfrak{m}$.) Proposition 7.9 implies that $C/\mathfrak{m} \cong k$. Now let x_i be image of t_i in $C/\mathfrak{m}$. Then, $f(x_1, \cdots, x_n) \neq 0$, since image of f under the map $A \to B \to C \to k$ is exactly $f(x_1, \cdots, x_n)$, i.e., evaluation map. but since f is converted to unit in C, so is $f(x_1, \cdots, x_n)$. However, $(x_1, \cdots, x_n) \in V$ since $\mathfrak{p}$ contains $r(\mathfrak{a})$, therefore the evaluation map gives zero for any polynomial in $\mathfrak{a}$.

15. *i)* $\implies$ *ii)*: By Proposition 2.19 iv), for any injective module map $f : M \to M'$, $M \bigotimes_A A \to M' \bigotimes_A A$ is just the same f, so A is flat A-module. And direct sum of flat module is flat by Exercise 2.4. So A^n is flat for any $n \in \mathbb{N}$.

    ii) $\implies$ *iii)*: Proposition 2.19 iv) implies it.

    iii) $\implies$ *iv)*: Since $m \bigotimes_A M \to A \bigotimes_A M$ is injective, this induces a short exact sequence

$$0 \to m \bigotimes_A M \to A \bigotimes_A M \to A/\mathfrak{m} \bigotimes_A M \to 0.$$

    Notes that $\operatorname{Tor}_1^A(A/\mathfrak{m}, M) := \ker(m \bigotimes_A M \to A \bigotimes_A M) = 0$ by definition of Tor.

$iv) \implies i)$: Let $x_1, \cdots, x_n$ be elements of M whose image in $M/\mathfrak{m}M$ form a k-basis of this vector space. By Proposition 2.8, the x_i generate M. Let F be a free A-module with basis $e_1, \cdots, e_n$ and define $\phi : F \to M$ by $\phi(e_i) = x_i$. Let $E = \ker(\phi)$. Then, the exact sequence

$$0 \to E \to F \to M \to 0$$

gives us an exact sequences

$$0 \to k \bigotimes_A E \to k \bigotimes_A F \xrightarrow{1\otimes\phi} k \bigotimes_A M \to 0.$$

Since $k = A/\mathfrak{m}A$, so $k \bigotimes_A F \cong (A/\mathfrak{m}M)^n$ and $k \bigotimes_A M \cong (M/\mathfrak{m}M)^n$ are vector space of the same dimension over $k = A/\mathfrak{m}A$, it follows that $1 \otimes \phi$ is an isomorphism (since in this case $1 \otimes \phi$ is a vector space map (as $A/\mathfrak{m}A$-module) thus surjectivity with the same dimension implies isomorphism.) Thus, $k \bigotimes_A E$ is zero dimensional k-vector space, thus 0. Notes that F is finitely generated A-module, thus F is Noetherian by Proposition 7.2. Thus, By Proposition 6.2, E is finitely generated since E is a submodule of F. Also, A is local ring, thus $\mathfrak{m}$ is Jacobson radical. And $E/\mathfrak{m}E = 0$ implies $\mathfrak{m}E = E$. Thus By Proposition 2.6 (Nakayama's lemma) $E = 0$. Thus, ϕ is isomorphism. Hence M is free.

16. $i) \implies ii)$: $A_\mathfrak{p}$ is a local ring and Noetherian by Corollary 7.4. Also, $M_\mathfrak{p}$ is still finitely generated as $A_\mathfrak{p}$-module, by identifying a generator x as $x/1$. Also, by Proposition 3.10, $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. Hence, by Exercise 7.15, $M_\mathfrak{p}$ is free $A_\mathfrak{p}$-module.

    $ii) \implies iii)$: Clear.

    $iii) \implies i)$: Free implies flat by Claim IX, and apply Proposition 3.10.

17. In a similar manner, define *irreducible module*; suppose $N \subseteq M$. Then N is irreducible module if there are no two proper submodules of M whose intersection is N. In other words, if there are two submodules L^1, L^2 whose intersection $L^1 \cap L^2$ is M, then $L^1 = M$ or $L^2 = M$.

    Lemma 7.11-1. *In a Noetherian module, every submodule is finite intersection of irreducible modules.*

    Proof. Suppose not. Then the set of submodules of N for which the lemma is false is not empty. Also, every chain of submodule in the set is stationary by Noetherian property of N. Thus by Zorn's lemma, the set has a maximal element, say M. Thus, M is reducible; otherwise M itself is a finite intersection of irreducilbe module. Thus, $M = L^1 \cap L^2$. Hence, each of L^1 and L^2 is a finite intersection of irreducible ideals, by maximality of M. Therefore so is M, contradiction. □

    Lemma 7.12-1. *In a Noetherian ring every irreducible module is primary.*

    Proof. By passing to the quotient module, it is enough to show that if the zero module is irreucible then it is primary. (Notes that M is Noetherian, then the quotient is Noetherian by an exact sequence $0 \to N \to M \to M/N \to 0$ and Proposition 6.3.) Let x be a zero divisor of N such that $xy = 0$ for some nonzero $y \in \mathbb{N}$. Then submodule $0 \subseteq \mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(x^2) \cdots$ form an ascending chain. By the a.c.c., this chain is stationary. We have $\mathrm{Ann}_M(x^n) = \mathrm{Ann}_M(x^{n+1})$ for some $n \in \mathbb{N}$. Thus, for any $a \in A$, $x(ay) = a(xy) = 0$. Also, if $ay \in x^nM$, then $ay = x^nb$ for some $b \in M$. Hence, $x^{n+1}b = x(x^nb) = xay = 0$. Thus, $b \in \mathrm{Ann}_M(x^{n+1}) = \mathrm{Ann}_M(x^n)$. Therefore, $ay = x^nb = 0$. This implies that $(y) \cap x^nM = 0$. Since (0) is irreducible submodule and $(y) \neq 0$, this implies $x^nM = 0$. Thus, x is nilpotent. By definition in Exercise 4.21, this implies that zero module is primary. □

18. $i) \implies ii)$: By Exercise 7.17, zero module has a primary decomposition. Let $N_1 \cap \cdots \cap N_k$ be an irredundant primary decomposition. (We can get irredundant decomposition by get rid of redundant one.) Then, without loss of generality, let $\mathfrak{p} := r_M(N_1)$. let $N_1^c = \bigcap_{j\neq 1}^k N_j$. It suffices to show that $\mathfrak{p} = \mathrm{Ann}(x)$ for some $x \in M$. From irredundant condition, $N_1^c \not\subseteq N_1$. Let $y \in N_1^c \setminus N_1$. Since A is Noetherian, and $r(N_1 : M) = \mathfrak{p}$, by Proposition 7.14, there exists $m \in \mathbb{N}$ such that $\mathfrak{p}^m \subseteq (N_1 : M)$. Hence, for given $y \in M$, there is $m \in \mathbb{N}$ such that $\mathfrak{p}^m y \subseteq N_1$. Suppose this m is the smallest one; i.e.,

$\mathfrak{p}^m \subseteq (N_1 : y)$ but $\mathfrak{p}^{m-1} \not\subseteq (N_1 : y)$. Then, $x \in N_1 \setminus \mathfrak{p}^{m-1}y$, such that $x = ay$ for some $a \in \mathfrak{p}^{m-1}$. Since $y \in N_1^c$, $\mathfrak{p}x \subseteq \mathfrak{p}^m y \subseteq N_1 \cap N_1^c = 0$. Hence, $\mathfrak{p} \subseteq \operatorname{Ann}_M(x)$.

Conversely, every elements in $(N_1 : x)$ is zero divisor of M/N_1. Since N_1 is primary, $(N_1 : x)$ is subset of nilpotent on M/N_1. Thus, $(N_1 : x) \subseteq \mathfrak{p}$. Hence, $\operatorname{Ann}_M(x) \supseteq (N_1 : x)$. This implies $\operatorname{Ann}_M(x) = (N_1 : x)$ since N_1 contains zero module. Thus, $\mathfrak{p} = \operatorname{Ann}_M(x)$, as desired.

ii) $\implies$ *iii)*: Notes that $\mathfrak{p} = \operatorname{Ann}(x) = \ker(A \to Ax)$ by definition of annihilator. Thus, $A/\mathfrak{p} = A/\operatorname{Ann}(x) \cong Ax \subseteq M$ by the first isomorphism theorem.

iii) $\implies$ *ii)*: Let $x \neq 0 \in N \cong A/\mathfrak{p}$, a submodule of M isomorphic to $A/\mathfrak{p}$. Now notes that $ax = 0$ if and only if $a \in \mathfrak{p}$ since x can be identified as a nonzero elements in $A/\mathfrak{p}$. Thus, $\operatorname{Ann}_M(x) = \mathfrak{p}$.

ii) $\implies$ *i)*: By Exercise 7.17, 0 is decomposable. By Theorem 4.5$*$ which we proved in the Exercise 4.22, all prime belong to 0 are all prime ideals in the set $\{r(\operatorname{Ann}(y)) : y \in M\}$. Since $\operatorname{Ann}(x) = \mathfrak{p}$, $r(\operatorname{Ann}(x)) = \mathfrak{p}$. Hence, $\mathfrak{p}$ belongs to 0.

For the last statement, start with the fact that 0 is decomposable by Exercise 7.17. Hence, fix a prime ideal $\mathfrak{p}$ belongs to 0, and take M_1 be the submodule of M isomorphic to $A/\mathfrak{p}$. Now, for M/M_1, which is Noetherian by Prooposition 6.3 with $0 \to M_1 \to M \to M/M_1 \to 0$, we can do the same thing to have M_2/M_1 such that $M_2/M_1 \cong A/\mathfrak{p}'$ for some $\mathfrak{p}'$ prime ideal. Thus this gives an ascending chain $0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$. Since M is Noetherian, this chain stabilizes at some point, say n. We claim that $M_n = M$. Otherwise, M/M_n is Noetherian by Proposition 6.3, thus 0 module in M/M_n is decomposable, thus we can generate M_{n+1} which is strictly greater than M_n in the same manner, contradiction. Thus, $M_n = M$.

19. Since ideal is a module, it suffices to show that when N is a submodule of a Noetherian module M with two minimal decompositions

$$N = \bigcap_{i=1}^{r} B_i = \bigcap_{j=1}^{s} C_i$$

then $r = s$ and that $r_M(B_i) = r_M(C_i)$ for all i (up to renumbering.)

If we assume the hint is true, then, let j_i be the number in $[s]$ such that

$$N = B_1 \cap \cdots \cap B_{i-1} \cap C_{j_i} \cap B_{i+1} \cap \cdots \cap B_r.$$

Notes that this one is also an irreducible decomposition. (We don't need minimality condition since the hint holds without such assumption, as we will see below). If $r < s$, then by applying the hint r times, we can get $N = \bigcap_{i=1}^{r} C_{j_i}$. This implies that $\bigcap_{j=1}^{s} C_i$ is not a minimal decomposition, contradiction. Thus, $r \geq s$. Conversely, if $r > s$, then we can apply the hint to replacing C_is with B_{j_i}s. $r > s$ also lead us to get contradiction with minimality of $\bigcap_{i=1}^{r} B_i$. Thus, $r = s$. Then, by exercise 4.22, $\{r_M(B_i)\}_{i=1}^{r} = \{r_M(C_i)\}_{i=1}^{r}$. After suitable renumbering (it is possible since it is finitely many), we may assume that $r_M(B_i) = r_M(C_i)$.

To show the hint, let $i \in [r]$ and let $B_i^c := \bigcap_{j\neq k}^{s} B_j$. Then, define $N_j = B_i^c \cap C_j$ for all $j = 1, \cdots, s$. Since each B_j contains N, $\bigcap_{j=1}^{s} N_j = (\bigcap_{j=1}^{s} C_i) \cap B_i^c = N$. Let $\pi_i : M \to B_i$ the canonical projection. Then, $\pi_i(\bigcap_{j=1}^{s} N_j) = \pi_i(N) = 0$ since N is submodule of B_i. Thus, $\bigcap_{j=1}^{s} \pi_i(N_j)$ form a decomposition of zero module in M/B_i. Moreover, since B_i is irreducible, zero module in M/B_i is irreducible; otherwise, using 1-1 correspondence between submodules of M/B_i and submodules of M containing B_i, we can show that B_i is reducible, contradiction. Hence, the decomposition $\bigcap_{j=1}^{s} \pi_i(N_j)$ implies that $\exists k$ such that $\pi_i(N_k) = 0$. This implies that $N_k \subseteq B_i$. Thus, $N_k \subseteq N$, which shows $N_k = N$ by construction. This is proof of hint.

20. (a) If E is a finite union of sets of the form $U \cap C$ where U is open and C is closed, then $E = U \cap C$ for some open and closed set (since finite union of open is open and finite union of closed is closed) and by definition $U, C \in \mathscr{F}$, so is E since $\mathscr{F}$ is closed under finite intersection. Conversely, notes that $\mathscr{F}$ contains all closed subsets, since it is closed under complement and has all open sets of X. By intersecting with X, which is clopen, we regard that closed sets and open sets are of form $U \cap C$ where U is open and C is closed. Now, all finite intersections of sets in $\mathscr{F}$ also of form

$U \cap C$ since $(U_1 \cap C_1) \cap (U_2 \cap C_2) = (U_1 \cap U_2) \cap (C_1 \cap C_2)$, which is intersection of open and closed sets. Thus, E should have a form $U \cap C$.

(b) If E contains a nonempty open set, then by Exercise 1.19, the open subset is dense, so is E. Conversely, suppose E is dense. Since $E = U \cap C$, this implies $X = \overline{U \cap C}$. From the definition of closed set, $\overline{C} = C$. By exercise 1.19, $\overline{U} = X$. Thus, $\overline{U \cap C} \subseteq \overline{U} \cap \overline{C} = X \cap C = C$. This implies $U \subseteq X = C$, thus $E = U \cap C = U$, hence E contains a nonempty open set.

21. Suppose $E \notin \mathscr{F}$. Then, $E \cap X = E$. Thus, if we let Σ be collection of closed sets C such that $C \cap E \notin \mathscr{F}$, then $X \in \Sigma$. By definition of Noetherian space, Σ has a minimal element, say X_0.

We claim that X_0 is irreducible. By claim II, it suffices to show that any two proper closed set of X_0 cover it. If C_1, C_2 are proper closed set cover X_0, then $(E \cap C_0), (E \cap C_1) \in \mathscr{F}$, thus $(E \cap C_0) \cap (E \cap C_1) = E \cap X_0 \in \mathscr{F}$, contradiction. Thus X_0 is irreducible.

If $\overline{E \cap X_0} \subsetneq X_0$, then by minimality of X_0, $E \cap \overline{E \cap X_0} \in \mathscr{F}$. However, we claim that $E \cap \overline{E \cap X_0} = E \cap X_0$. Notes that $E \cap \overline{E \cap X_0} \supseteq E \cap X_0$ is clear. Let $x \in E \cap \overline{E \cap X_0}$. Then, $x \in E$, so we need to show that $x \in X_0$. Since $x \in \overline{E \cap x_0}$, every open neighborhood U of x contains an element $y \in E \cap X_0$. This implies that U meets X_0 for every open neighborhood U. Thus, $x \in \overline{X_0}$. Since x is already in E, $x \in E \cap \overline{X_0} = E \cap X_0$ since X_0 is closed set. Therefore, $E \cap X_0 = E \cap \overline{E \cap X_0} \in \mathscr{F}$ gives contradiction.

On the other hand, assume that $E \cap X_0$ contains a nonempty open subset of X_0, say $U_0 = U \cap X_0$, for some open set U of X. Then, $U_0 \in \mathscr{F}$ by construction, and $U^c \cap X_0$ contains $E \cap X_0 \setminus U_0 = E \cap X_0 \cap U_0^c = E \cap X_0 \cap (U \cap X_0)^c = E \cap X_0 \cap (U^c \cup X_0^c) = E \cap X_0 \cap U^c$. (Also notes that $U^c \cap X_0 \in \mathscr{F}$ since it is closed.) This implies that

$$E \cap X_0 = U_0 \cup (E \cap X_0 \setminus U_0) = (U \cap X_0) \cup (E \cap X_0 \cap U^c).$$

By Exercise 7.20 i), $E \cap X_0$ is finite union of sets of form $U \cap C$, thus $E \cap X_0 \in \mathscr{F}$, contradiction.

Thus by contrapositive, if RHS holds then $E \in \mathscr{F}$.

Conversely, if $E \in \mathscr{F}$, then either E contains a nonempty open set of X_0 or E is not dense in X_0, by Exercise 7.20 ii). This is just restatement of RHS.

22. If E is open in X, then $E \cap X_0$ is open in X_0. Thus, by Exercise 1.19, either $E \cap X_0$ is empty or dense in X_0. Convesely, suppose that E is not open. Then, $E \cap X = E$ is not open. Let Σ be collection of closed set C such that $E \cap C$ is not open in C. Since Σ is nonempty, and X is Noetherian, there is a minimal element, say X_0. Moreover, by the same argument as in the proof of Exercise 7.21, X_0 is irreducible. Notes that $E \cap X_0 \neq \emptyset$ since $\emptyset$ is open in X_0. Suppose that $E \cap X_0$ contains a nonempty open subset $U_0 = U \cap X_0$ of X_0 for some open set U of X. Let $C_0 = X_0 - U_0 = U_0^c \cap X_0 = U^c \cap X_0$. Then, $C_0 \cap E$ is open in C by minimality of X_0. Hence, $C_0 \cap E = U_1 \cap C_0 = U_1$ for some open set U_1 of X. Then,

$$E \cap X_0 = (U_0 \cap E \cap X_0) \cup (C_0 \cap E) = (U_0 \cap X_0) \cup (U_1 \cap U^c \cap X_0) = (U_0 \cup U_1 \cap U^c) \cap X_0 = (U_0 \cap U_1 \cup U_0 \cap U^c) \cap X_0 = U_0 \cap U_1 \cap X_0.$$

Thus, $E \cap X_0$ is open in X_0, contradiction.

23. By Exercise 7.20, it suffices to take $E = U \cap C$ where U is open and C is closed in Y. Then $C = V(\mathfrak{a})$ for some radical ideal $\mathfrak{a}$, thus, by replacing B with $B/\mathfrak{a}$, using the homeomophism (Exercise 1.21 iv))$\operatorname{Spec}(B/\mathfrak{a}) \cong V(\mathfrak{a})$ and homomorphism $A \to B \to B/\mathfrak{a}$, we may assume that E is open in Y. Notes that Y is Noetherian by Corollary 7.7. Hence, E is quasi-compact by Exercise 6.6. Thus E is covered by finite union of basic open sets Y_g for some $g \in B$. By Exercise 3.21, Y_g is homeomorphic to $\operatorname{Spec}(B_g)$. Thus, E is covered by finite union of basic open sets of the form $\operatorname{Spec}(B_g)$. If we show that each image of Y_g is constructible, then image of E is constructible since $\operatorname{Im}(E) = \operatorname{Im}(Y_g \cup \cdots) = \operatorname{Im}(Y_g) \cup \cdots$, and collection of constructible sets are closed under finite union. Thus, we can just assume that $E = Y$, by replacing B with B_g using $A \to B \to B_g$.

Let X_0 be an irreducible closed subset of X such that $f^*(Y) \cap X_0$ is dense in X_0. (If $f^*(Y) \cap X_0$ is not dense in X_0, then nothing to prove since it is a condition in Exercise 7.21.) By Exercise 1.20, $X_0 = V(\mathfrak{p}) \cong \operatorname{Spec}(A/\mathfrak{p})$ for some minimal prime $\mathfrak{p}$ in X. Then, $f^*(Y) \cap X_0 = f^*(f^{*-1}(X_0))$; to see this, let $\mathfrak{p}' \in f^*(Y) \cap X_0$. Then, $\exists \mathfrak{q} \in Y$ such that $f^{\mathfrak{q}} \in X_0$. Thus, $\mathfrak{q} \in f^{*-1}(X_0)$, therefore $\mathfrak{p}' \in f^*(f^{*-1}(X_0))$. Conversely, if $\mathfrak{p}' \in f^*(f^{*-1}(X_0))$, then $\mathfrak{p}' \in X_0$. Also, $\mathfrak{p}'$ is in the image of Y, thus $\mathfrak{p}' \in f^*(Y) \cap X_0$.

Moreover,

$$f^{*-1}(X_0) = f^{*-1}(V(\mathfrak{p})) \underbrace{=}_{Exercise 1.21 ii)} V(\mathfrak{p}^e) \underbrace{=}_{Exercise 1.21 iv)} \operatorname{Spec}(B/\mathfrak{p}^e) \cong \operatorname{Spec}(A/\mathfrak{p} \bigotimes_A B)$$

(To see last equality, notes that $a(1 \otimes 1) \in A/\mathfrak{p} \bigotimes_A B$ is equal to $(a \otimes 1)$ and $(1 \otimes f(a))$. Thus, $a \in \mathfrak{p}$ iff $a(1 \otimes 1) = 0$ iff $(1 \otimes f(a)) = 0$. Also, if $b \in \mathfrak{p}^e$, then $b = \sum_{j=1}^n b_i f(a_i)$ for all $a_i \in \mathfrak{p}$, thus $1 \otimes b_i f(a_i) = a_i(1 \otimes b_i) = (a_i \otimes b_i) = (0 \otimes b_i) = 0$. This shows the equality.

From this equality, we can see that the restriction of f as $f : A/\mathfrak{p} \to A/\mathfrak{p} \bigotimes_A B \cong B/\mathfrak{p}$ which induces $f^* : \operatorname{Spec}(A/\mathfrak{p} \bigotimes_A B) = V(\mathfrak{p}^e) \to V(\mathfrak{p}) = \operatorname{Spec}(A/\mathfrak{p})$. Since $f^*(\operatorname{Spec}(A/\mathfrak{p} \bigotimes_A B)) = f^*(f^{*-1}(X_0)) = f^*(Y) \cap X_0$ is dense by assumption, Exercise 1.21 v) says that $\ker(f) \subseteq \mathfrak{R}$, a nilradical. Since $A/\mathfrak{p}$ is integral domain, this implies $\mathfrak{R} = 0$, thus f is injective. map.

Now we replace A with $A/\mathfrak{p}$, and B with $B/\mathfrak{p}^e$. Then, we just assume that $f : A \to B$ is injective and A is integral domain. If $Y_1, \cdots, Y_n$ are irreducible components of Y, it is enough to show that some $f^*(Y_i)$ contains a nonempty open sets in X, to use criterion in Exercise 7.21. Since X is assumed to be X_0, which is irreducible, and $f^*(Y)$ is assumed to be dense in $X = X_0$. Thus, $X = \overline{f^*(Y)} = \overline{\bigcup_{j=1}^n f^*(Y_j)} = \bigcup_{j=1}^n \overline{f^*(Y_j)}$ since it is closure of finite union. Since X is irreducible, there is a fixed j such that $\overline{f^*(Y_j)} = X$, thus $f^*(Y_j)$ is dense in X. Hence, by applying Exercise 1.21 with the fact that $Y_j = V(\mathfrak{b})$ for some minimal prime of B (from Exercise 1.20), we can think that $A \to B \to B/\mathfrak{b}$ is still injective. Thus, by replacing B with $B/\mathfrak{b}$ (thus Y with Y_j), we may assume that B is integral domain and f is injective. Now it suffices to show that $f^*(Y)$ contains an open subset of X.

Since A, B are integral domain, and B is finitely generated A-algebra, (from f is of finite type,) thus B is finitely generated $f(A)$-algebra. Thus, by replacing A with $f(A)$, A can be regarded as a subring of B. Then, we can use Exercise 5.21 to assume that there is $s \neq 0$ in A such that $g : A \to \Omega$ is a homeomorphism for which $f(s) \neq 0$, then g can be extended to a homomorphism $B \to \Omega$, for any algebraically closed field Ω. Now let $\mathfrak{p}$ be a prime ideal not containing s. (Such prime ideal exists, since nilradical as a intersection of all prime ideal is zero in the integral domain.) Then we have a map $g : A \to A/\mathfrak{p} \to k(\mathfrak{p}) \to \Omega$ where $k(\mathfrak{p})$ is a field of fraction of $A/\mathfrak{p}$ and Ω is algebraically closure of $k(\mathfrak{p})$. Then, since $g(s) \neq 0$, by the Exercise 5.21, this g can be extended to $B \to \Omega$. Let $\mathfrak{q} = \ker(B \to \Omega)$. Then, $\mathfrak{q} \cap A = \mathfrak{p}$ since it is just restriction of $B \to \Omega$ over A. Thus, $\mathfrak{p} \in f^*(Y)$, as an image of $\mathfrak{q}$. Thus, $X_s \subseteq f^*(Y)$. Done.

Hence, for any irreducible closed set X_0 of X, $f^*(E) \cap X_0$ is not dense, or if it is dense, then it contain an open set. Therefore, Exercise 7.21 shows that $f^*(E)$ is constructible.

24. If f^* is open map, then Exercise 5.10 says it has going down property. Conversely, suppose f has the going-down property. Let Y_s be a basic open set of Y for some $s \in B$. Then, $Y_s \cong \operatorname{Spec}(B_s)$ by Exercise 3.21. Hence, by replacing B with B_s, we may assume that $E = f^*(Y)$ is open in X. By the going down property, if $\mathfrak{p}$ is prime ideal of B such that $\mathfrak{p} \cap f(A) = \mathfrak{q}$ and $\mathfrak{q} \supseteq \mathfrak{q}'$, then there exists $\mathfrak{p}' \in \operatorname{Spec}(B)$ such that $\mathfrak{p} \supseteq \mathfrak{p}'$ and $\mathfrak{p}' \cap f(A) = \mathfrak{q}'$. In other words, if $\mathfrak{q} \in E = f^*(Y)$ such that $\mathfrak{q} \subseteq \mathfrak{q}'$, then $\mathfrak{q}' \in E$.

    Now, to use Exercise 7.22, let X_0 be arbitrary irreducible closed subset of X and X_0 meets E. Then, if $\mathfrak{q} \in E \cap X_0$ then every prime ideal contained in $\mathfrak{q}$ lies in E by going down property. By Exercise 1.20 iv), $X_0 = V(\mathfrak{p})$ for some minimal prime $\mathfrak{p}$ of X. This shows that $\mathfrak{q}$ contains $\mathfrak{p}$, thus $\mathfrak{p} \in E \cap X_0$. Moreover, $\overline{\{\mathfrak{p}\}} = X_0$ by Exercise 1.18 ii). Thus, $X_0 \supseteq \overline{E \cap X_0} \supseteq V(\mathfrak{p}) = X_0$ implies $E \cap X_0$ is dense in X_0. Since $E = f^*(Y)$ is constructible by Exercise 7.23, and X_0 is closed thus constructible, so is $E \cap X_0$. By Exercise 7.20 ii), since $E \cap X_0$ is constructible and dense in X_0, $E \cap X_0$ should contain a nonempty open set of X_0. Thus, by Exercise 7.22, E is open.

25. By Exercise 5.11, f has going down property. By Exercise 7.24, f^* is open map.

26. In modern notation, let M be a module, and (M) be a isomorphism classes of module M, and $C(A) = \mathbb{Z}F(A)$. $D(A)$ is a subgroup generated by $\{(M') - (M) + (M'') : 0 \to M' \to M \to M'' \to 0 \text{ is exact}\}$. $K(A) = C(A)/D(A)$. $\gamma(M) = (M) + D$ in $K(A)$.

   (a) Let $\lambda : C(A) \to G$. Then, if $0 \to M' \to M \to M'' \to 0$ is exact sequence, then $\lambda(M) = \lambda(M') + \lambda(M'')$, i.e.,
$$\lambda(M') - \lambda(M) + \lambda(M'') = 0.$$
   Thus, λ sends $D(A)$ to 0. Thus, define $\lambda_0(\gamma(M)) = \lambda(M)$. It is well-defined since $\lambda(D(A)) = 0$. And this is unique; if there is λ_0' such that $\lambda(M) = \lambda_0'(\gamma(M))$, then λ_0 agrees with λ_0' for all $\gamma(M)$. However, these are all elements of $K(A)$. Thus, $\lambda_0 = \lambda_0'$.

   (b) From Exercise 7.18, for any fixed M, there exists a chain $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$. Thus, $0 \to M_{i-1} \to M_i \to A/\mathfrak{p}_i \to 0$ is a short exact sequence. And $M_1 \cong A/\mathfrak{p}_1$. Thus, $M_i = \bigoplus_{j=1}^{i} A/\mathfrak{p}_j$. Therefore, $K(A)$ is generated by $\gamma(A/\mathfrak{p})$ for all prime ideal $\mathfrak{p}$.

   (c) If A is a PID, then for any principal ideal (a) with $a \neq 0$, we have $0 \to A \to A \to A/(a) \to 0$ where $A \to A$ is sending x to ax. This implies $\gamma(A) - \gamma(A) + \gamma(A/(a)) = 0$, thus $\gamma(A/(a)) = 0$ for all $a \neq 0$. Thus, the only nonzero elements of generators of $K(A)$ is $\gamma(A/(0)) = \gamma(A)$ since (0) is prime in the Noetherian ring. (PID is Noetherian; since every submodule (ideals) are finitely generated.). Thus, $K(A)$ is an abelian group generated by a single element $\gamma(A)$. Now from the structure theorem of PID, A^n is not the same as direct sum of $A/(a)$ for any ideal (a). Hence, $\gamma(A^n)$ is nonzero. Thus, $K(A)$ is infinite cyclic group generated by $\gamma(A)$, which is isomorphic to $\mathbb{Z}$.

   (d) Notes that every finitely generated B-module is finitely generated A-module; let M be generated by $x_1, \cdots, x_n$ and B is generated by $y_1, \cdots, y_m$ as A-module. (That's the meaning of f is finite.) Then, $M = \{\sum_{i=1}^{n} b_i m_i : b_i \in B\} = \{\sum_{i=1}^{m} \sum_{j=1}^{n} a_i y_i x_j : a_i \in A\}$. Thus M is finitely generated by $x_i y_j, (i,j) \in [n] \times [m]$.
   Also, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated B-modules, then it is also exact sequence of finitely generated A-modules, since such a change of view doesn't change each maps, especially kernel and images. Thus, if we let $\iota : F(B) \to F(A)$, then this induces a map $\gamma_A \circ \iota : F(B) \to F(A) \to K(A)$. Hence, by the universal property of i), $\exists f_! : K(B) \to K(A)$ such that $f_!(\gamma_B(N) = \gamma_A \circ \iota(N)$ for any B-module ismorphism classes (N). If $g : B \to C$ is another finite ring homomorphism, then by this argument, we have $g_! : K(C) \to K(B)$ such that $g_!(\gamma_C(N)) = \gamma_B(N)$. Hence, $f_! \circ g_!(\gamma_C(N)) = \gamma_A(N)$. Conversely, if we construct $(g \circ f)_!$, then $(g \circ f)_!(\gamma_C(N) = \gamma_A(N)$. Thus, $f_! \circ g_!$ and $(g \circ f)_!$ agrees on all $\gamma_C(N)$ for all N. Since those are all elements of $K(C)$, $f_! \circ g_! = (g \circ f)_!$.

27. (a) Assume that we already give an free abelian group structure on $F_1(A)$. First of all, if M is generated by x_i for $i \in [n]$ and N is generated by y_j for $j \in [m]$, then $M \bigotimes_A N$ is generated by $x_i \otimes y_j$ for $(i,j) \in [n] \times [m]$. Thus, $M \bigotimes_A N$ is finitely generated. Moreover, by Exercise 2.8.i), $M \bigotimes_A N$ is flat if M, N are flat. Moreover, if $M \cong M', N \cong N'$ then $M \bigotimes_A N \cong M' \bigotimes_A N'$ by [3][p.27]. Hence, we can define tensor product of isomorphism classes as well. Since it is associative and binary, $F_1(A)$ with tensor product is a multiplicative monoid, since it has identity (A) as a flat module. (We already know that direct sum of flat module is flat; see Exercise 2.4.) Also, since tensor product of flat module over an exact sequence is still exact. This gives a distribution law on $F_1(A)$.Thus, $F_1(A)$ is abelian group with respect to addition (direct sum), multiplication by tensor product is associative and distributive over addition, and commutative by Propostion 2.14 i), and it has multiplicative identity (A). Hence, $F_1(A)$ is a commutative ring. Moreover, $D_1(A)$ is ideal; since for any generator of $D_1(A)$, say $0 \to M \to N \to L \to 0$ exact, tensor with any flat module over this exact sequence is still exact (thus in the generator of D_1), thus again it

is element of $D_1(A)$. Moreover, $D_1(A)$ has already abelian group structure inherited from original definition. Hence, $D_1(A)$ is ideal of $F_1(A)$. Thus, $K_1 = F_1(A)/D_1(A)$ is also a commutative ring. Notes that we can do the same thing on $F(A)$ and $K(A)$, thus $K(A)$ is a commutative ring containing $K_1(A)$ as a subring.

(b) Notes that tensor product is distributive over direct sum, and it is associative, and tensor with identity doesn't change anything. Thus, it satisfies module axioms. Thus $K(A)$ is $K_1(A)$-module by tensoring.

(c) Suppose A is local ring. Then, by Exercise 7.15, every flat module is free. Thus, $F_1(A)$ is a set of all free modules over A. Thus, $F_1(A)$ is an infinite cyclic group generated by the isomorphism class (A). Thus, $F_1(A) \cong \mathbb{Z}$.

(d) By Exercise 2.20 [3][p.29], if M is flat and finitely generated A-module, then $B \bigotimes_A M$ is flat and finitely generated B-module. Thus, we can define a map $g : F(A) \to F(B)$ by sending M to $B \bigotimes_A M$. (It is well-defined since isomorphism is preserved under tensor.) Moreover,

Claim XLIX. $B \bigotimes_A (-)$ *preserves exact sequence*

Proof. To see this, notes that if $f : N \to M$ is injective as an A-module map, then $B \bigotimes_A N \to B \bigotimes_A M$ is injective; to see this, let $b \otimes n$ in the kernel. Then, $b = 0$ or $f(n) = 0$, i.e., $n = 0$ since f is injective. Thus, $\ker(B \bigotimes_A N \to B \bigotimes_A M) = 0 \bigotimes_A N \cup B \bigotimes_A 0 = \{0 \otimes 0\}$ by definition of tensor product. Also, surjectivity is preserved; let $f : N \to M$ be a surjective map. Then, for any $b \otimes m$ in $B \bigotimes_A M$, from surjectivity of f, $\exists n \in \mathbb{N}$ such that $f(n) = M$, hence $1 \otimes f(b \otimes n) = b \otimes m$.

Now let $0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0$ is an exact sequence. Then, $1 \otimes g \circ 1 \otimes f(b \otimes n) = 1 \otimes g(b \otimes f(n)) = b \otimes g(f(n)) = b \otimes 0 = 0$. Thus, $\ker 1 \otimes g \supseteq \operatorname{Im} 1 \otimes f$. Conversely, let $b \otimes m \in \ker 1 \otimes g$. Then, $b \otimes g(m) = 0$. This implies $b = 0$ or $m \in \ker g$. If $b = 0$, then for any $n \in \mathbb{N}$, $1 \otimes f(0 \otimes n) = 0 \otimes f(n) = 0 \otimes m$ since both are identified with $0(0 \otimes f(n)) = 0 \otimes 0 = 0(0 \otimes m)$. If $b \neq 0$, then $m \in \ker(g) = \operatorname{Im} f$, thus $\exists n \in N$ such that $f(n) = m$, therefore $1 \otimes g(b \otimes m) = b \otimes g(m) = b \otimes g(f(n)) = b \otimes 0 = 0$. Hence, $\ker 1 \otimes g \subseteq \operatorname{Im} 1 \otimes f$, done. ☐

Thus, $\gamma_B \circ g$ is additive map. Therefore, $\gamma_B \circ g : F(A) \to F(B) \to K(B)$ induces a map $f^! : K(A) \to K(B)$ by the universal property in Exercise 7.26 i) such that

$$f^! \circ \gamma_A(M) = \gamma_B \circ g(M).$$

If $h : B \to C$ is another ring homomorphism with C is Noetherian, then we have a map $h^! : K(B) \to K(A)$ such that $h^! \circ \gamma_B(M) = \gamma_C(C \bigotimes_B M)$. Thus, $h^! \circ f^!(\gamma_A(M)) = h^!(\gamma_B \circ (B \bigotimes_A M)) = \gamma_C(C \bigotimes_B (B \bigotimes_A M))$. Now, by apply this universal property on $h \circ f$, we get

$$(h \circ f)^! \circ \gamma_A(M) = \gamma_C \circ C \bigotimes_A M.$$

Now, notes that $C \bigotimes_B (B \bigotimes_A M) \underbrace{\cong}_{Exercise 2.15} (C \bigotimes_B B) \bigotimes_A M \cong C \bigotimes_A M$. Thus, $(h \circ f)^! = h^! \circ f^!$.

(e) Since f is finite ring homomophism, Exercise 7.26 iv) gives $f_!$. Then, let $\gamma_A(M) \in K_1(A), \gamma_B(N) \in K(B)$. Then, $f^!(\gamma_A(M)) = \gamma_B(B \bigotimes_A M)$, Now by defined in ii), $f^!(\gamma_A(M))\gamma_B(N) = \gamma_B(B \bigotimes_A M)\gamma_B(N) = \gamma_B((B \bigotimes_A M) \bigotimes_B N)$. Thus,

$$f_!(f^!(\gamma_A(M))\gamma_B(N)) = f_!(\gamma_B((B \bigotimes_A M) \bigotimes_B N)) = \gamma_A(B \bigotimes_A M) \bigotimes_B N).$$

where $(B \bigotimes_A M) \bigotimes_B N$ is regarded as an A-module. Also,

$$\gamma_A(M) f_!(\gamma_B(N)) = \gamma_A(M)\gamma_A(N) = \gamma_A(M \bigotimes_A N)$$

by regarding N as an A-module. Now notes that

$$(B \bigotimes_A M) \bigotimes_B N \underbrace{=}_{\text{Proposition 2.14 i)}} (M \bigotimes_A B) \bigotimes_B N \underbrace{=}_{\text{Exercise 2.15}} M \bigotimes_A (B \bigotimes_B N) = M \bigotimes_A N$$

by regarding N as an A-module. Thus, $f_!(f^!(\gamma_A(M))\gamma_B(N)) = \gamma_A(M) f_!(\gamma_B(N))$ as desired.
In other words, regarding $K(B)$ as a $K_1(A)$ module by restriction of scalars, the homomorphism $f_! : K(B) \to K(A)$ is a $K_1(A)$-module homomorphism. (The above equation is just showing a condition of module map, by acting $K_1(A)$ on $K(B)$ via $f^!$.)

8 Artin Rings

1. For fixed i, Proposition 7.14 implies that $\exists r_i \in \mathbb{N}$ such that $\mathfrak{p}_i^{r_i} \subseteq \mathfrak{q}_i$. Then, by Theorem 4.12$*$ 6,

$$\mathfrak{p}_i^{(r_i)} \underbrace{=}_{\text{Definition}} S_{\mathfrak{p}_i}(\mathfrak{p}_i^{r_i}) \underbrace{\subseteq}_{\text{Theorem 4.12}*\text{, 6}} S_{\mathfrak{p}_i}(\mathfrak{q}_i) \underbrace{=}_{\text{Theorem 4.12}*\text{, 3}} \mathfrak{q}_i$$

   where Theorem 4.12$*$ in [4].

   Suppose $\mathfrak{q}_i$ is an isolated primary component. Then $\mathfrak{p}_i$ is a minimal prime. Notes that $A_{\mathfrak{p}_i}$ is a Noetherian local ring by Proposition 7.3. By Corollary 3.13, $\mathfrak{p}_i A_{\mathfrak{p}_i}$ is also minimal. Since $\mathfrak{m} := \mathfrak{p}_i A_{\mathfrak{p}_i}$ is maximal by construction, this implies that $A_{\mathfrak{p}_i}$ has only one prime ideal and Noetherian. Thus, by Theorem 8.5, $A_{\mathfrak{p}_i}$ is Artinian. Moreover, $\mathfrak{m}$ is both nilradical and Jacobson radical. By Proposition 8.4, $\mathfrak{m}$ is nilpotent. This implies that $\exists r \in \mathbb{N}$ such that $\mathfrak{m}^r = 0$.

   Honestly, I don't know how $\mathfrak{m}^r = 0$ implies $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all large r. Instead, I refer [4]. Notes that $\mathfrak{p}^{(r_i)} \cap \bigcap_{j \neq i} \mathfrak{q}_j \subseteq \bigcap_{j=1}^n \mathfrak{q}_j = 0$ is another primary decomposition of 0 since $\mathfrak{p}^{(r_i)}$ is $\mathfrak{p}$-primary by Exercise 4.13 i). By Corollary 4.11, isolated primary component is unique; thus $\mathfrak{p}^{(r_i)} = \mathfrak{q}_i$. (It holds for all $r > r_i$.)

   Instead, if $\mathfrak{q}_i$ is an embedded primary component, then $\mathfrak{p}_i$ is not a minimal prime ideal, thus $\mathfrak{p}_i A_{\mathfrak{p}_i}$ contains a prime ideal of $A_{\mathfrak{p}_i}$ by Corollary 3.13. Thus, $A_{\mathfrak{p}_i}$ is not Artinian, but Noetherian local ring. By Proposition 8.6, $(\mathfrak{p}_i A_{\mathfrak{p}_i})^r$ are all distinct. Thus, its contraction $\mathfrak{p}_i)^{(r)}$ are all distinct. Hence in the given primary decomposition, since $\mathfrak{p}_i)^{(r)} \subseteq \mathfrak{q}_i$ for all $r \geq r_i$, we can replace $\mathfrak{q}_i$ by any of the infinite set of $\mathfrak{p}_i$-primary ideals $\mathfrak{p}_i)^{(r)}$ where $r \geq r_i$, and so there are infinitely many minimal primary decompositions of 0 which differ only in the $\mathfrak{p}_i$-components.

2. $i) \implies ii)$: A has only finitely many prime ideals which are all maximal by Proposition 8.1, 8.3. This implies $\operatorname{Spec}(A)$ is finite. Also, it implies that all singleton in $\operatorname{Spec}(A)$ is closed. Hence, each singleton is open because $\operatorname{Spec}(A)$ minus one element is a finite union of closed sets so closed, hence its complement, which is a singleton, is open. Thus, $\operatorname{Spec}(A)$ has discrete topology.

   $ii) \implies iii)$: Clear.

   $iii) \implies i)$: In this case, A is Noetherian ring where every prime ideal is maximal. Thus, $\dim A = 0$. Hence, by Theorem 8.5, A is Artinian.

3. By Theorem 8.7, A is product of Artin local ring. For each component B of the product, $A \to B$ is a projection map, thus by sending generator of A as an k-algebra, we can conlude that B is also finitely generated k-algebra. Thus, if we show this statement for all Artin local ring, then, A itself is a finite k-algebra since it is product of finite k-algebra, thus again finite. (Generator of A as finitely generated k-module is just collection of generators of all components of the product.)

Thus, assume A is Artin local ring. Then, it has the maximal ideal $\mathfrak{m}$, thus $A/\mathfrak{m}$ is a field, which is finitely generated over k (by sending generators of A as k-algebra to $A/\mathfrak{m}$. By Corollary 7.10, $A/\mathfrak{m}$ is a finite algebraic extension of k, i.e., $\dim_k(A/\mathfrak{m}) < \infty$. Notes that A is Artinian by given condition i), and Noetherian since it is finitely generated k-algebra and Corollary 7.7. Thus, A is Noetherian A-module, hence by Exercise 7.18 iii), there is a chain of submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = \mathfrak{m}$ such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of A. Since A is Artin local ring, all $\mathfrak{p}_i = \mathfrak{m}$. Hence, $0 \to M_i \to M_{i+1} \to A/\mathfrak{m} \to 0$ is exact sequence. Thus, as a k-module (i.e. k-vector space), these are still exact since A-module map can be viewed as a k-linear transform. Thus, $\dim_k M_i + \dim_k(A/\mathfrak{m}) = \dim_k M_{i+1}$. Thus, $\dim_k \mathfrak{m} = \dim M_r = r\dim_k(A/\mathfrak{m}) < \infty$. Lastly, we have an exact sequence $0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0$, and this also can be viewed as k-module exactness, therefore $\dim_k A = \dim_k \mathfrak{m} + \dim_k(A/\mathfrak{m}) = (r+1)\dim_k(A/\mathfrak{m}) < \infty$. Hence A is finite k-algebra.

ii) $\implies$ *i)*: Since A is finite k-algebra, A is finitely generated k-module, i.e., k-vector space. Thus, by Proposition 6.10 iv) A satisfy d.c.c. thus Artinian.

4. *i)* $\implies$ *iii)* $\implies$ *ii)*: By Exercise 3.21 iv), for each $\mathfrak{p} \in \mathrm{Spec}(A)$, $f^{*-1}(\mathfrak{p}) \cong \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \bigotimes_A B)$. Since *i)* implies B is a finitely generated as an A-module, assume that $B = \{\sum_{i=1}^n a_i b_i : a_i \in A\}$. Then, every element in $k(\mathfrak{p}) \bigotimes_A B$ is of form $\sum_{j=1}^n k_i \otimes a_i b_i = \sum_{j=1}^n \overline{a_i} k_i \otimes b_i = \sum_{j=1}^n k_i' \otimes b_i$. Thus, $k(\mathfrak{p}) \bigotimes_A B$ is a finitely generated $k(\mathfrak{p})$-algebra, and dimension as a $k(\mathfrak{p})$-vector space is n.

(In detail, first of all, generate $k(\mathfrak{p})^n$ and give a map sending b_i to e_i in $k(\mathfrak{p})^n$, where $\{e_i\}$is the standard basis of $k(\mathfrak{p})^n$. This is $k(\mathfrak{p})$-vector space map since it preserves scalar multiple (over $k(\mathfrak{p})$) and summation. Moreover, it is surjective, hence injective as a vector space. Thus, it is isomorphic as a $k(\mathfrak{p})$-module. Also, multiple in this ring is defined as $(k\otimes b)\cdot(k'\otimes b) = kk'\otimes bb'$ as in [3][p.30]. And since B is an algebra, we have $b_i b_j = \sum_{k=1}^n a_{ijk} b_k$ for some $a_{ijk} \in A$. Thus, for any $k \in k(\mathfrak{p}), k'\otimes b_i, k''\otimes b_j \in k(\mathfrak{p}) \bigotimes_A B$,

$$\begin{aligned}
k((k'\otimes b_i)\cdot(k''\otimes b_j)) &= k(\sum_{l=1}^n k'k''\overline{a_{ijl}} \otimes b_l) = kk'k''\sum_{l=1}^n \overline{a_{ijl}} \otimes b_l. \\
(k(k'\otimes b_i))\cdot(k''\otimes b_j) &= (\sum_{l=1}^n kk'k''\overline{a_{ijl}} \otimes b_l) = kk'k''\sum_{l=1}^n \overline{a_{ijl}} \otimes b_l. \\
(k'\otimes b_i)\cdot(k(k''\otimes b_j)) &= (\sum_{l=1}^n kk'k''\overline{a_{ijl}} \otimes b_l) = kk'k''\sum_{l=1}^n \overline{a_{ijl}} \otimes b_l.
\end{aligned}$$

Thus, multiple is $k(\mathfrak{p})$-bilinear. Therefore, $k(\mathfrak{p}) \bigotimes_A B$ is a finite k-algebra, which is iii); Exercise 8.3 implies that $k(\mathfrak{p}) \bigotimes_A B$ is Artinian. Exercise 8.2 implies that $\mathrm{Spec}(k(\mathfrak{p}) \bigotimes_A B)$ is discrete. Thus $f^{*-1}(\mathfrak{p}) \cong \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \bigotimes_A B)$ is discrete, which is ii);

iii) $\implies$ *ii)*: By Exercise 3.21 iv), $\mathrm{Spec}(B \bigotimes_A k(\mathfrak{p})) \cong f^{*-1}(\mathfrak{p})$. By the same argument using Exercise 8.3 and 8.2, it is discret subspace of $\mathrm{Spec}(B)$.

iii) $\implies$ *iv)*: By Exercise 3.21 iv), $\mathrm{Spec}(B \bigotimes_A k(\mathfrak{p})) \cong f^{*-1}(\mathfrak{p})$. By iii), we know that $B \bigotimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra. Hence, by Exercise 3), $B \bigotimes_A k(\mathfrak{p})$ is Artinian. By Exercise 2 ii), $\mathrm{Spec}(B \bigotimes_A k(\mathfrak{p}))$ is finite.

For the last question, since f may not be a finite type, we cannot use the Remark in [3][p.60] (In this case, f is of finite, thus fibres of f^* are finite by this Exercise.) For the counterexample if f is not a finite type, I refer [4]; let k be a field which is not algebraically closed, and think $\overline{k}$, an algebraic closure of k. Then, $f : k \to \overline{k}$ is a canonical injection. Which leads to $f^* : \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$. Since

$\mathrm{Spec}(\overline{k})$ and $\mathrm{Spec}(k)$ are both singleton, thus they are finite, and f^* is also finite map. (It maps zero to zero, since $f^{-1}(0) = (0)$ because f is injection.) However, f is not finite, i.e., $\overline{k}$ is not finitely generated k-vector space unless $\dim_k(\overline{k}) < \infty$. And as you can see, if $k = \mathbb{R}$, then $\overline{k} = \mathbb{C}$, which is 2-dimensional $\mathbb{R}$-vector space, thus it is finite. However, if $k = \mathbb{Q}$, then $\overline{Q}$ is not finite $\mathbb{Q}$-vector space, since $[\mathbb{Q} : \sqrt[n]{2}] = n$ for any $n \in \mathbb{N}$ and $\overline{Q}$ contains $\sqrt[n]{2}$ for any $n \in \mathbb{N}$. It holds for $\mathbb{F}_p, \mathbb{Q}_p, k(t)$ where t is transcendental element, and so on.

5. From Exercise 5.16, there is a linear map $\pi : k^n \to k^r$ such that $\pi|_X : X \to L$ is surjective, where X is an affine variety of k^n and $L = k^r$ as a subspace of k^n (thus $r < n$.) Let B be a coordinate ring of L, which is isomorphic to $k[y_1, \cdots, y_r]$ by conclusion of Exercise 5.16. Then the lemma says that A is integral over B, and the map $\phi : B \to A$ is given by $f \mapsto f \circ \pi|_X$. Thus, we have $\phi^* : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$. Now for given $l \in L$, $\mathfrak{m}_l$ is the maximal ideal in B consisting of all regular functions which are zero at l, whose existence was shown in the Exercise 1.27. Then, think $\phi^{*-1}(\mathfrak{m}_l)$; Suppose that $\mathfrak{p} \in \mathrm{Spec}(A)$ is in $\phi^{*-1}(\mathfrak{m}_l)$. Then for any $f \in \mathfrak{p}$, $\phi(f) \in \mathfrak{m}_l$, which implies $f \circ \pi|_X \in \mathfrak{m}_l$, which implies that for any $x \in X$ such that $\pi|_X(x) = l$, $f \circ \pi|_X(x) = 0$. This implies that $f \in \mathfrak{m}_x$ for all $x \in \pi|_X^{-1}(l)$. Thus, definitely, $\mathfrak{m}_x \in \phi^{*-1}(\mathfrak{m}_l)$ for all $x \in \pi|_X^{-1}(l)$. Thus, it suffices to show that the fibres of ϕ^* is finite; in that case $\phi^{*-1}(\mathfrak{m}_l)$ is finite for any l, thus there are only finitely many $x \in X$ such that $\pi|_X(x) = l$ (otherwise, $\phi^{*-1}(\mathfrak{m}_l)$ contains infinitely many $\mathfrak{m}_x$s, contradiction.) To use the Exercise 8.4, we need to show that $\phi : B \to A$ is a finite map, i.e., we should show that ϕ makes A be a finitely generated B-algebra.

   And, since A is integral over B, and if we identify B with $\phi(B)$, a subring of A, then $B[\overline{x_i} : i = 1, \cdots, n] = A$. To see this, notes that A is still finitely generated as a k-algebra over $\overline{x_i}$s, and B still contains k in its subring. Thus, A is a finitely generated B-module by Corollary 5.2, since A is integral over B implies $\overline{x_i}$ is integral over B. Thus, $\phi : B \to A$ is finite since A is finitely generated $f(B)$-algebra, which means just A is a finitely generated B algebra, since we define A as a B module acting by multiplication through $f(B)$. Hence, by Exercise 8.4, fibers of ϕ^* are finite. Thus, $\pi|_X^{-1}(l)$ is finite.

   To see that $\max_{l \in L} \pi|_X^{-1}(l)$ is bounded, notes that $A = B[\overline{x_i} : i = 1, \cdots, n]$. Then, for any $\mathfrak{p} \in \mathrm{Spec}(B)$, Exercise 3.21 iv) shows that $\mathrm{Spec}(A \bigotimes_B k(\mathfrak{p})) \cong \phi^{*-1}(\mathfrak{p})$. Since A is finitely generated by $x_i, i = 1, \cdots, n$, as we've shown in the proof of Exercise 8.4 $i) \implies iii) \implies ii)$, $\dim_{k(\mathfrak{p})} A \bigotimes_B k(\mathfrak{p}) = n$. Now Exercise 8.3. implies that $A \bigotimes_B k(\mathfrak{p})$ is Artinian, thus Exercise 8.2 ii) implies that $\mathrm{Spec}(A \bigotimes_B k(\mathfrak{p}))$ is discrete and finite. Thus, let $\mathrm{Spec}(A \bigotimes_B k(\mathfrak{p})) = \{\mathfrak{m}_i\}_{i=1}^m$ for some $m \in \mathbb{N}$. Since every prime ideals in this set is maximal, they are coprime, therefore, $A \bigotimes_B k(\mathfrak{p}) \to \prod_{i=1}^m (A \bigotimes_B k(\mathfrak{p}))/\mathfrak{m}_i$ is a surjective ring homomorphism by Proposition 1.10. Since $A \bigotimes_B k(\mathfrak{p})$ is finite $k(\mathfrak{p})$-algebra, so is $(A \bigotimes_B k(\mathfrak{p}))/\mathfrak{p}$ by sending generators of $A \bigotimes_B k(\mathfrak{p})$, hence $(A \bigotimes_B k(\mathfrak{p}))/\mathfrak{p}$ is a $k(\mathfrak{p})$-vector space, as $(A \bigotimes_B k(\mathfrak{p}))/\mathfrak{p}$ is a $k(\mathfrak{p})$-vector space. Since the surjective ring homomorphism $A \bigotimes_B k(\mathfrak{p}) \to \prod_{i=1}^m (A \bigotimes_B k(\mathfrak{p}))/\mathfrak{m}_i$ is definitely a $k(\mathfrak{p})$-algebra map therefore a $k(\mathfrak{p})$ module map. This implies that the surjective map is actually $k(\mathfrak{p})$-linear map, therefore surjectivity implies that $\dim_k(\prod_{i=1}^m (A \bigotimes_B k(\mathfrak{p}))/\mathfrak{m}_i) \le n$. Since $A \bigotimes_B k(\mathfrak{p}))/\mathfrak{m}_i$ is nonzero vector space, m is at most n. This shows that $|\mathrm{Spec}(A \bigotimes_B k(\mathfrak{p}))| \le n$, as desired. Thus all fibres of ϕ^* is bounded.

6. Let $\mathfrak{q} = \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_n = \mathfrak{p}$ be a chain of primary ideal. By taking radical on the chain we assume that each $\mathfrak{a}_i$ are $\mathfrak{p}$-primary. By Noetherian condition, $n < \infty$. Now to see such chain is bounded, suppose that for each $n \in \mathbb{N}$ there is a chain $\mathfrak{q} = \mathfrak{a}_{n1} \subseteq \cdots \subseteq \mathfrak{a}_{nn} = \mathfrak{p}$. Then, for any two chain $\mathfrak{q} = \mathfrak{a}_{n1} \subseteq \cdots \subseteq \mathfrak{a}_{nn} = \mathfrak{p}$ and $\mathfrak{q} = \mathfrak{a}_{m1} \subseteq \cdots \subseteq \mathfrak{a}_{mm} = \mathfrak{p}$, we can make a chain by intersection, such as

$$\mathfrak{q} = \mathfrak{a}_{m1} \cap \mathfrak{a}_{n1} \subseteq \mathfrak{a}_{m1} \cap \mathfrak{a}_{n2} \cdots \mathfrak{a}_{m1} \cap \mathfrak{a}_{nn} \subseteq \mathfrak{a}_{m2} \cap \mathfrak{a}_{n2} \subseteq \cdots \mathfrak{a}_{mm} \cap \mathfrak{nn} = \mathfrak{p}.$$

This new chain contain the original length n chain and m chain, thus has length at least m-chain. Thus, we can assume that the original length n chain can be extended to length m-chain for any $m \in \mathbb{N}$. Thus, by this construction we can make a infinite chain by intersecting all such length n chain with $n \in \mathbb{N}$, contradiction. Thus, the length of chain must be bounded.

Now, if we have a maximal chain C which is not the greatest length, then by above intersection with a chain with the greatest length chain, we can refine the C having length n, as the above argument shows, which is the contradiction of maximality. Thus, all maximal chains have the same length.

(I think [4]'s proof is a little bit suspicious since we don't know whether primary ideals of A are 1-1 corresponds to primary ideals of $(A/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}$; at least Proposition 3.9 doesn't say anything on that matter, and neither do other propositions in Section 3.)

9 Discrete Valuation Rings and Dedekind Domains

In [3][p.94], if $\mathfrak{a} \neq 0$ is an ideal in A, there is a least integer k such that $v(x) = k$ for some $x \in \mathfrak{a}$. By surjectivity of valuation map, $\exists x_1 \in A$ such that $v(x_1) = 1$. Thus, $v(xx_1^n) = k + n$. Now, for any $y \in A$ such that $v(y) = k + n$ with $k \geq 0$, $(y) = (xx_1)$ by the previous paragraph of [3]. Hence, $y \in \mathfrak{a}$. This implies that the only ideals $\neq 0$ in A are the ideals $\mathfrak{m}_k = \{y \in A : v(y) \geq k\}$.

In the proof of Proposition 9.2, notes that given condition implies (A) in [3][p.95] since Noetherian local dimension 1 ring has (0) as its prime ideal, thus all nonzero prime ideals are maximal, and locality implies that there is only one nonzero prime ideal. And since Noetherian, a minimal primary decomposition of $\mathfrak{a}$ consists of only one ideal which is $\mathfrak{m}$-primary, thus $\mathfrak{a}$ is $\mathfrak{m}$-primary. For (B) see that otherwise A is Artinian by Proposition 8.6. Theorem 8.5 says that $\dim A = 0$, contradicting assumption that $\dim A = 1$.

In the proof of ii) to iii) of Proposition 9.2, if $x^{-1}\mathfrak{m} \subseteq \mathfrak{m}$, then $\mathrm{Ann}_{A[x^{-1}]}(\mathfrak{m}) = 0$. To see this, notes that $A\mathfrak{m} = \mathfrak{m}$ implies $\mathrm{Ann}_{A[x^{-1}]}(\mathfrak{m}) \cap A = 0$. Also, $x^{-1}\mathfrak{m} \neq 0$ since $x^{-1} = b/a$ thus $ax^{-1} = b \neq 0$. (Otherwise $b \in (a)$.) Thus $\mathfrak{m}$ is a faithful $A[x^{-1}]$-module. And it is finitely generated as an A-module since A is Noetherian. However, $x^{-1}\mathfrak{m} \subseteq A$; to see this, since $b \in \mathfrak{m}^{n-1}$, $b\mathfrak{m} \subseteq \mathfrak{m}^n \subseteq \mathfrak{a}$. Thus, $b/a\mathfrak{m} \subseteq 1/a\mathfrak{m}^n \subseteq 1/a(a) \subseteq A$.

In the proof of iii) to iv) of Proposition 9.2, if $\mathfrak{m} = (x)$, then $\overline{x}$ is still a generator of $\mathfrak{m}/\mathfrak{m}^2$. Thus, it form a basis by Converse of Proposition 2.8. And this converse holds since $\overline{x}$ still generates $M/\mathfrak{m}M = \mathfrak{m}/\mathfrak{m}^2$, thus any element in $\mathfrak{m}/\mathfrak{m}^2$ is of form $a\overline{x}$ for any $a \in A/\mathfrak{m}$, thus $\mathfrak{m}/\mathfrak{m}^2$ is a vector space generated by a basis $\{\overline{x}\}$. If $\overline{x} = 0$ then it is 0 dimensional. However, by (B), it is nonzero; thus $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.

In the proof of iv) to v) of Proposition 9.2, by iv), $\mathfrak{m}$ is principal, by Proposition 2.8. Thus let $\mathfrak{m} = (b)$. Suppose n is the least integer such that $\mathfrak{a} \supseteq \mathfrak{m}^n$ holds. Then, $b^{n-1} \notin \mathfrak{a}$ but $b^n \in \mathfrak{a}$. By proposition 8.8 applying on $A/\mathfrak{m}^n$, we know that $\overline{\mathfrak{a}}$ is principal, thus $\mathfrak{a}$ is principal. Say $\mathfrak{a} = (a)$. Then, $b^n = ap$ for some $p \in A$ but $a = bq$ for some $q \in A$. This implies $b^n = bpq$. Since A is domain, this implies $b^{n-1} = pq$. Since $\mathfrak{m}$ is prime, $p \in \mathfrak{m}$ or $q \in \mathfrak{m}$. This shows that $p = bp'$. Then, $b^{n-2} = p'q$. Repeating this argument, we may conclude that $p = b^k$, $q = b^{k'}$ for some $k + k' = n - 1$. Thus, $a = bq = b^{k'+1}$. This implies that $\mathfrak{a}$ is a power of $\mathfrak{m}$.

In the proof of theorem 9.5, the author says that any nonzero prime ideal $\mathfrak{p}$ of A is maximal. To see this in detail, notes that if $\mathfrak{p}^c = \mathfrak{p} \cap \mathbb{Z}$ is zero, then since $(0) \subseteq \mathfrak{p}$ and (0) is prime ideal since A is an integral domain (since it is subset of Q) Corollary 5.9 says that $\mathfrak{p} = (0)$, contradiction. Hence, $\mathfrak{p} \cap \mathbb{Z} \neq 0$. Since $\mathbb{Z}$ is PID, every prime ideal is maximal by Example 3 after (1.6). Thus, by Corollary 5.8, $\mathfrak{p}$ is also maximal in A. In conclusion, A is Noetherian domain of dimension one, and integrally closed. Thus, by Theorem 9.3, A is Dedekind Domain.

In the proof of Proposition 9.7 $\implies$ part, $M = (x^r/y)$ since $yM = (x^r)$. Let $v(y) = s$. Then, y, x^s have the same value, yx^{-s} has value 0, thus unit, thus in A (by definition of valuation ring.) Thus, $(x^r/y) = (x^{r-s})$.

In the other part, $\mathfrak{m}^{-1}\mathfrak{a} \supseteq \mathfrak{a}$ since $\mathfrak{m}^{-1}\mathfrak{a}$ is an integral ideal, thus for any $a/m \in \mathfrak{m}^{-1}\mathfrak{a}$, $a/m \cdot m = a$ also lies in $\mathfrak{m}^{-1}\mathfrak{a}$. Also notes that in the local domain, $\mathfrak{m}$ is Jacobson radical.

1. From A is integrally closed, $S^{-1}A$ is also integrally closed by Proposition 5.12. By Proposition 7.3, $S^{-1}A$ is Noetherian. Moreover, since $S^{-1}A \subseteq K$, a field of fraction of A, $S^{-1}A$ is domain. Since dimension of A is 1, dimension of $S^{-1}A$ is 0 or 1, by Proposition 3.11 iv). If $\dim S^{-1}A$ is 0, then every prime ideal except 0 meets S. Thus $S^{-1}A$ is field, since (0) is the only prime ideal. Otherwise, there

exists at least one nonzero prime ideal of A not meeting S. In summary, $S^{-1}A$ is Noetherian local domain with dimension 1 which is integrally closed. Then, by Theorem 9.3, it is Dedekind domain.

For the second statement, we need a lemma. (We cannot directly use Corollary 3.15 since $(A : M)$ is defined over K not A.)

Claim L ([9] Lemma 25.4). *Let A be domain, K be a field of fraction of A, and let M, N be fractional ideal, and let S be a multiplicative subset of A. Then, $S^{-1}MN = (S^{-1}M)(S^{-1}N)$ and $S^{-1}(M : N) \subseteq (S^{-1}M : S^{-1}N)$ with equality if N is finitely generated.*

Also, if M is invertible, then $S^{-1}M^{-1} = S^{-1}(A : M) = (S^{-1}A : S^{-1}M)$.

Proof. Given $x \in S^{-1}(MN)$, $x = \frac{\sum_{j=1}^{k} m_j n_j}{s}$ for some $m_j \in M, n_j \in N, s \in S, k \in \mathbb{N}$. Then, $x = \sum_{j=1}^{k} \frac{m_j}{1} \frac{n_j}{s} \in (S^{-1})(S^{-1}N)$. Conversely, if $x \in (S^{-1}M)(S^{-1}N)$, then $x = \sum_{j=1}^{k} \frac{m_j}{s_j} \frac{n_j}{t_j}$. Let $q = \prod_{j=1}^{k} s_j t_j$, and $o'_i = m_i n_i \prod_{j \neq i}^{k} s_j t_j$, we have $x = \sum_{j=1}^{k} \frac{m_j}{s_j} \frac{n_j}{t_j} = \sum_{j=1}^{k} \frac{o'_i}{q}$ with $o'_i \in MN$. Thus $x \in S^{-1}(MN)$.

Also, if $z \in S^{-1}(M : N)$, then $z = x/s$ for $x \in (M : N)$ and $s \in S$. For any $y \in S^{-1}N$, $y = n/t$ for some $n \in N, t \in S$. Then $zy = xn/st$ with $xn \in M$, $st \in St$, thus, $zS^{-1}N \subseteq S^{-1}M$. Hence, $z \in (S^{-1}M : S^{-1}N)$. Conversely, if N is finitely generated, say $n_1, \cdots, n_r$. Then given $z \in (S^{-1}M : S^{-1}N)$, write $zn_i/1 = m_i/s_i$ with $m_i \in M, s_i \in S$. Let $s = \prod s_i$. Then, $szn_i \in M$ for all i. Thus, $sz \in (M : N)$. Thus, $z = sz/s \in S^{-1}(M : N)$, as desired.

Finally, if M is invertible, then $S^{-1}(A) = S^{-1}(M(A : M)) = (S^{-1}M)(S^{-1}(A : M)) \subseteq (S^{-1}M)(S^{-1}A : S^{-1}M) = S^{-1}A$ shows that $S^{-1}(A : M)$ and $(S^{-1}A : S^{-1}M)$ are two inverse of $S^{-1}M$. Hence, by usual argument about inverse, they are equal. $\square$

By Proposition 3.11 i), $\phi : \mathfrak{a} \mapsto S^{-1}\mathfrak{a}$ is well-defined surjective map from $I(A)$ to $I(S^{-1}A)$. Since A is Noetherian, every fractional ideal is finitely generated. Thus by the above claim it is multiplicative homomorphism, since it preserves multiplication and inverse element. Moreover, $\phi(P(A)) \subseteq P(S^{-1}A)$ since for any principal fractional ideal (u) with $u \in K$, $S^{-1}(u) := \{u(a/s) : a/s \in S^{-1}A\}$, thus principal. Thus, we can define $\phi' : H \to H'$ by $\overline{M} \mapsto \overline{\phi(M)}$, where $P = P(A), P' = P(S^{-1}A)$. To see it is well-defined, let M and N are two distinct representation of $\overline{M}$. Then, $M = NQ$ for some principal fractional ideal Q. Then, $S^{-1}M = (S^{-1}N)S^{-1}Q$ by the above claim, thus $\overline{S^{-1}M} = \overline{S^{-1}N}$ in H'. Moreover, ϕ' is surjective since ϕ is surjective.

2. To use the hint "Localize at each maximal ideal" we need a lemma.

Claim LI. *Let M, N be two A-submodules of an A-module K. Module equality is local property; i.e., $M = N$ if and only if $M_\mathfrak{p} = N_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$*

Proof. If $M = N$, then trivial. Conversely, suppose $M_\mathfrak{p} = N_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$. Notes that M contains N iff $(N + M)/M = 0$. Then, we have an exact sequence

$$0 \to M \to M + N \to (M + N)/M \to 0,$$

and tensor with $A_\mathfrak{p}$. Since $A_\mathfrak{p}$ is flat module, we have an exact sequence

$$0 \to M_\mathfrak{p} \to M_\mathfrak{p} + N_\mathfrak{p} \to (M_\mathfrak{p} + N_\mathfrak{p}/M_\mathfrak{p}) \to 0.$$

By given condition, $M_\mathfrak{p} + N_\mathfrak{p} = M_\mathfrak{p}$ since they are the same, thus $(M_\mathfrak{p} + N_\mathfrak{p}/M_\mathfrak{p}) = 0$. Notes that $((M + N)/N)_\mathfrak{p} \cong (M_\mathfrak{p} + N_\mathfrak{p}/M_\mathfrak{p})$ by applying Corollary 3.4 iii) and 3.4 i). Thus, $((M + N)/N)_\mathfrak{p} = 0$ for all prime ideal $\mathfrak{p}$. By Proposition 3.8, this implies $(M + N)/N = 0$. Thus $N \subseteq M$. By the same argument over N, we have $M \subseteq N$. Thus $M = N$. $\square$

Hence, to see $c(fg) = c(f)c(g)$, it suffices to show that they are equivalent on $A/\mathfrak{m}$ for all maximal ideal $\mathfrak{m}$. Thus, we may assume that A be a localization of Dedekind domain by maximal ideal (since $\overline{c(f)}$ in $A/\mathfrak{m}$ can be seen as $c(\overline{f})$ where $\overline{f} = \overline{a_0} + \cdots + \overline{a_n}x^n$.)

Thus, we may assume that A is DVR. Let $g = b_0 + \cdots + b_m x^m$ in $A[x]$. Then, $c(fg) \subseteq c(f)c(g)$ since any coefficients in fg is generated by linear combination of product $a_i b_j$. To see $c(fg) \supseteq c(f)c(g)$, suppose $\mathfrak{m} = (y)$ for some $y \in A$. Since A is DVR, $c(f) = (y^t), c(g) = (y^s)$ for some $s, t \in \mathbb{N}$. Hence, $v(y) = 1$ and $\min_{i \in [n]} v(a_i) = t, \min_{i \in [m]} v(b_i) = s$. (We use property on [3][p.94].) Now let i', j' be element in $\mathbb{N}$ such that $v(a_{i'}) = t, v(b_{j'}) = s$ and for any $0 \le i < i', 0 \le j < j'$, $v(a_i) > s, v(b_j) > t$. Then, the coefficient of $x^{i'+j'}$ in fg is $\sum_{i+j=i'+j'} a_i b_j$, and $v(\sum_{i+j=i'+j'} a_i b_j) = \min_{i+j=i'+j'}(v(a_i) + v(b_j)) = s + t$ since either $i < i'$ or $j < j'$ holds. Thus, one of the generator of $c(fg)$ has value $s+t$. Moreover, every other generators of $c(fg)$ has value greater than or equal to $s+t$. This implies that $c(fg) \supseteq (y^{s+t}) = c(f)c(g)$ since $(\sum_{i+j=i'+j'} a_i b_j)A = y^{s+t}A$ by property on [3][p.94].

3. By the argument on [3][p.94], DVR is Noetherian. Conversely, suppose a valuation ring A is Noetherian. We claim that A is PID. Then, A is Noetherian (by given condition) local (by Proposition 5.18) domain (by definition of valuation ring) of dimension one having maximal ideal which is principal (since A is PID). Thus Proposition 9.2 shows that A is DVR.

   To see A is PID, let $\mathfrak{a}$ be an ideal. Since A is Noetherian, $\mathfrak{a}$ is finitely generated. Thus let $\mathfrak{a} = (a_1, \cdots, a_n)$. Then, for any $i, j \in [n]$, either $(a_i) \supseteq (a_j)$ or $(a_i) \subseteq (a_j)$ by Exercise 5.28. Thus, using inclusion as an order for $\{a_1, \cdots, a_n\}$, this gives total ordering. Since $\{a_1, \cdots, a_n\}$ is finite, there should be a maximal element a_i, i.e., (a_i) contains all (a_j). Then, $(a_i) \supseteq \mathfrak{a}$ implies $\mathfrak{a} = (a_i)$. So A is PID.

4. Let $\mathfrak{m} = (x)$ since $\mathfrak{m}$ is principal. We claim that every element y can be expressed as $y = ux^t$ with unique t. To see this, if y is unit, then by setting $u = y$, done. Suppose y is nonunit. Then, from the condition $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$, $\exists t \in \mathbb{N}$ such that $y \in \mathfrak{m}^k$ for all $k \le t$ but not in t.(Notes that if $y \in \mathfrak{m}^t$ then for any $k \le t$, $y \in \mathfrak{m}^k$ since $\mathfrak{m}^k \supseteq \mathfrak{m}^t$.) Thus if $y = vx^s$ for some $s < t$, with unit v, then $vx^s = ux^t$. Since it is domain, x is cancellable, thus $vu^{-1} = x^{t-s}$. This implies that x^{t-s} is unit, thus $\mathfrak{m} = (1)$, which implies $x = 1$, contradiction. Hence t is uniquely determined.

   Now let $v(y) = t$ for given expression $y = ux^t$ with unique t. Notes that this map $v : A \to \mathbb{N}$ is onto, and satisfies $v(yz) = v(y) + v(z)$ and $v(y+z) \ge \min(v(y), v(z))$. For the first one, since $y = ux^t$ and $z = vx^s$, $v(yz) = v(uvx^{t+s}) = t + s$. For the latter, if we assume $s < t$, then $y + z = x^s(ux^{t-s} + v)$. Thus, $v(y+z) = s + v(ux^{t-s} + v) \ge \min(v(y), v(z))$.

   Now, we can extend v from K^* to $\mathbb{Z}$, by defining $v(y/z) = v(y) - v(z)$. Then, since any element in K^* has expression ux^t with $t \in \mathbb{Z}$, and also satisfy the above properties. Hence, v is a discrete valution on K, and we observes that $A = \{x \in K : v(x) \ge 0\}$. Hence, A is valuation ring of v, thus A is DVR.

5. Let M be a finitely generated by $x_1, \cdots, x_n$. Then by the surjective homomorphism $A^n \to M$, $M \cong A^n/K$ for some submodule K of A^n. By Exercise 3.13, M is torsion free if and only if $M_{\mathfrak{p}}$ is torsion free for all prime ideals $\mathfrak{p}$.

   Suppose M is flat. Then, by Exercise 7.16, $M_{\mathfrak{p}}$ is free $A_{\mathfrak{p}}$ module. Thus, $M_{\mathfrak{p}} = A_{\mathfrak{p}}^{n_p}$ for some $n_p \in \mathbb{N}$. Since $M \cong A^n/K$, $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}^n/K_{\mathfrak{p}}$ by Corollary 3.4 iii) and Proposition 3.11 v). Thus, $A_{\mathfrak{p}}^{n_p} \cong A_{\mathfrak{p}}^n/K_{\mathfrak{p}}$. If $K_{\mathfrak{p}} \neq 0$, then $A_{\mathfrak{p}}^n/K_{\mathfrak{p}}$ is not a free module, since for given basis $\{e_i\}_{i=1}^n$ of $A_{\mathfrak{p}}^n$, an element in K can be expressed as $\sum_{i=1}^n a_i e_i$, thus $\sum_{i=1}^n a_i e_i = 0$ in $A_{\mathfrak{p}}^n/K_{\mathfrak{p}}$, hence $\{e_i\}_{i=1}^n$ is not linearly independent. Thus, for any generator of $M_{\mathfrak{p}}$, its set of generators in $M_{\mathfrak{p}} = A_{\mathfrak{p}}^n/K_{\mathfrak{p}}$ is not linearly independent. Thus, $M_{\mathfrak{p}}$ is free module as $A_{\mathfrak{p}}^{n_p}$ but not free as $A_{\mathfrak{p}}^n/K_{\mathfrak{p}}$, contradiction. This shows that $K_{\mathfrak{p}} = 0$, thus $n_p = n$ by Exercise 2.11. Hence, $K_{\mathfrak{p}} = 0$ for all prime ideal $\mathfrak{p}$. By Proposition 3.8, $K = 0$. Hence, M is free. By below claim M is torsion free.

   Claim LII. *If A is integral domain, M is free module, then M is torsion free.*

   Proof. Let $\{e_i\}_{i=1}^n$ be a basis of M. Any element in M can be expressed as $\sum_{i=1}^n a_i e_i$ for some $a_i \in A$. Now let $\sum_{i=1}^n a_i e_i$ be a torsion. Then by definition, $\exists b \neq 0 \in A$ such that $b \sum_{i=1}^n a_i e_i = 0$. Since e_is are linearly independent, $ba_i = 0$. Since A is domain, $ba_i = 0$ implies $a_i = 0$ for all i. Thus, $\sum_{i=1}^n a_i e_i = 0$. Hence M is torsion free. $\square$

Conversely, if M is torsion free, then $M_\mathfrak{p}$ is torsion free, since if m/s is torsion in $M_\mathfrak{p}$, then $\exists b \in A$ such that $bm/s = 0/1$, thus $tbm = 0$ for some $t \in S$, and since M is torsion free, this implies $m = 0$ or $tb = 0$. If m is zero, then m/s is not a torsion, if m is nonzero, then $tb = 0$, thus $b = 0$ since $0 \notin A - \mathfrak{p}$. Note that $A_\mathfrak{p}$ is PID since A is Dedekind domain, thus $A_\mathfrak{p}$ is DVR, and every ideal in DVR is power of the maximal ideal, which is principal, thus every ideal is principal. Now, from the structure theorem of PID, $M_\mathfrak{p}$ is direct sum of its free part and its torsion part. Since torsion part is zero, so $M_\mathfrak{p}$ is free. By Exercise 7.16, M is flat.

6. Notes that $M_\mathfrak{p}$ is still consists of all torsion elements, by the structure theorem of PID, $M_\mathfrak{p} \cong \bigoplus_{j=1}^{n_p} A_\mathfrak{p}/(d_j)$ for some $d_j \in A_\mathfrak{p}$. (We already show that $A_\mathfrak{p}$ is PID since it is DVR in the proof of the above exercise.) Since A is DVR, each $(d_j) = \mathfrak{p}^{k_j} A_\mathfrak{p}$ for some $k_j \in \mathbb{N}$, thus

$$M_\mathfrak{p} \cong \bigoplus_{j=1}^{n_p} A_\mathfrak{p}/(d_j) = \bigoplus_{j=1}^{n_p} A_\mathfrak{p}/\mathfrak{p}^{k_j} A_\mathfrak{p}.$$

Now we claim that there are only finitely many $\mathfrak{p}$ such that $M_\mathfrak{p}$ is nonzero. Since M is torsion submodule, $\mathrm{Ann}(M) \neq 0$. By Corollary 9.4, $\mathrm{Ann}(M) = \prod_{j=1}^{m} \mathfrak{p}_j^{n_j}$ with $n_j > 0$. Thus, $\mathrm{Supp}(M) = V(\mathrm{Ann}(M)) = \{\mathfrak{p}_j\}_{j=1}^{m}$ by Exercise 3.19 v). Now define a map $\phi : M \to \bigoplus_{j=1}^{n} M_{\mathfrak{p}_j}$ by $m \mapsto m/1$. Since A is Dedekind domain, every nonzero prime ideal is maximal. Thus, any two prime ideals are coprime. Thus, for given $i \neq j$, $\exists x \in \mathfrak{p}_i \setminus \mathfrak{p}_j$. Now let q be maximum of k_j in $M_{\mathfrak{p}_i}$ for all $i = 1, \cdots, ms$. Then, x^q annihilates $M_{\mathfrak{p}_j}$. Thus, $(M_{\mathfrak{p}_i})_{\mathfrak{p}_j} = 0$ since $x^n \in A - \mathfrak{p}_j$. However,

$$(M_\mathfrak{p})_\mathfrak{p} \underbrace{\cong}_{Proposition 3.5} (M \bigotimes_A A_\mathfrak{p}) \bigotimes_A A_\mathfrak{p} \underbrace{\cong}_{Exercise 2.15} M \bigotimes_A (A_\mathfrak{p} \bigotimes_A A_\mathfrak{p}) \underbrace{\cong}_{Proposition 3.5} M \bigotimes_A (A_\mathfrak{p})_\mathfrak{p}.$$

If we let $S = A - \mathfrak{p}$, then $(A_\mathfrak{p})_\mathfrak{p} = S^{-1}(S^{-1}A) = (SS)^{-1}A = S^{-1}A$ by Exercise 3.3 since $SS = S$ by multiplicative closedness of S. Thus, $(M_\mathfrak{p})_\mathfrak{p} \cong M \bigotimes_A (A_\mathfrak{p})_\mathfrak{p} \cong M \bigotimes_A A_\mathfrak{p} = M_\mathfrak{p}$. Thus, the given map ϕ is locally isomorphism, i.e., for any prime ideal $\mathfrak{p}$, $\phi_\mathfrak{p}$ is isomorphism. Hence by Proposition 3.9, ϕ is isomorphism.

Not it remains that show $A_\mathfrak{p}/\mathfrak{p}^k A_\mathfrak{p} \cong A/\mathfrak{p}^k$. To see this, let $\psi : A/\mathfrak{p}^k \to A_\mathfrak{p}/\mathfrak{p}^k A_\mathfrak{p}$ by $\overline{x} \mapsto \overline{x/1}$. It is injective, since if $\overline{x/1} = 0$, then $x/1 \in \mathfrak{p}^k A_\mathfrak{p}$. This implies that $x/1 = y/s$ for some $y \in \mathfrak{p}^k, s \in A - \mathfrak{p}$. Hence, $\exists t \in A - \mathfrak{p}$ such that $tsx = ty$. Since A is domain, $tsx = ty$ implies $sx = y$. By Corollary 9.4, (sx) has unique factorization as a product of prime ideals. Since $sx \in \mathfrak{p}^k$, $(sx) \subseteq \mathfrak{p}^k$. However $(sx) = (s)(x)$ and $(s) \notin \mathfrak{p}$. Thus, (sx) is product of factorization of (s) and (x), and we know that (s) has no factor about $\mathfrak{p}$. Since this factorization must include $\mathfrak{p}^k$ or higher than k part, this implies that (x) has factorization $\mathfrak{p}^k$ or higher part. Thus, $x \in \mathfrak{p}^k$. Thus, $\overline{x} = 0$. This shows that ψ is injective.

Also, to see ψ is surjective, let $\overline{x/s} \in A_\mathfrak{p}/\mathfrak{p}^k A_\mathfrak{p}$. Notes that $\mathfrak{p}$ is maximal since A is Dedekind domain. And $s \in A - \mathfrak{p}$. Thus, in $A/\mathfrak{m}$ image of s is nonzero and unit, thus $\exists y \in A - \mathfrak{p}$ such that image of sy is 1. This implies that $sy - 1 \in \mathfrak{p}$. Thus, $(sy-1)^k \in \mathfrak{p}^k$. Thus, expand $(sy-1)^k$ as $(sy-1)^k = 1 - sp$ for some $p \in A$, then, p (or $-p$) is inverse of s in $A/\mathfrak{p}^k$. Thus, $\overline{x/s} = \psi(\overline{xp})$ since $xp/1 = x/s$ by $xsp = x$. Thus ψ is surjective. Thus, $A_\mathfrak{p}/\mathfrak{p}^k A_\mathfrak{p} \cong A/\mathfrak{p}^k$ for any prime ideal $\mathfrak{p}$.

7. From the proof of above exercise, $A_\mathfrak{p}/\mathfrak{p}^k A_\mathfrak{p} \cong A/\mathfrak{p}^k$. Thus for any $\mathfrak{a} = \mathfrak{p}^k$, since $A_\mathfrak{p}$ is DVR, so every ideal in $A_\mathfrak{p}$ principal, so is $A_\mathfrak{p}/\mathfrak{p}^k A_\mathfrak{p}$, thus so is $A/\mathfrak{p}^k$. For general ideal $\mathfrak{a}$, Corollary 9.4, says that $\mathfrak{a} = \prod_{j=1}^{n} \mathfrak{p}_j^{n_j}$. Thus, define a projection $\phi : A \to \prod_{j=1}^{n} A/\mathfrak{p}_j^{n_j}$. Then, $\ker(\phi) = \bigcap \mathfrak{p}_j^{n_j}$, By 1.10, since each pair of primes is coprime, so is pair of power of primes, thus $\mathfrak{a} = \ker(\phi)$. Thus $A/\mathfrak{a} \cong \prod_{j=1}^{n} A/\mathfrak{p}_j^{n_j}$. Now we claim that product of PID is PID. Notes that by definition of product, every ideal is actually product of each ideals in each component. Since each ideal in the component is pid, say generated by b_j, so the ideal in the product is generated by $(b_1, \cdots, b_n)$, thus PID. Hence, $A/\mathfrak{a}$ is PID since contraction of the ideal is PID and every ideal in $A/\mathfrak{a}$ is contraction of the ideal in the product.

To see every ideal is generated by at most 2 elements, let $\mathfrak{b}$ be an ideal which is not principal. Let $a \in \mathfrak{b}$. Then, $\mathfrak{b} \supsetneq (a)$, thus $\mathfrak{b}/(a)$ is nonzero ideal in $A/(a)$ by 1-1 correspondence (Proposition 1.1.) Since $A/(a)$ is principal, $\mathfrak{b}/(a)$ is generated by one element, say $b + (a)$. Then, we claim that $(a, b) = \mathfrak{b}$. Notes that $\phi : A \to A/(a)$, $\phi(a, b) = (b + (a))$. Thus, 1-1 correspondence implies that $(a, b) = \mathfrak{b}$.

8. As we've shown above, module equality is local property. Also, by Proposition 3.11 v), sum and intersection commutes with localization. Thus we may assume that A is DVR. Then, $\mathfrak{m}$ is principal, say $\mathfrak{m} = (x)$. And any ideal $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are $(x^a), (x^b), (x^c)$ for some $a, b, c \in \mathbb{N}$. Also notes that sum of (x^n) and (x^m) is $(x^{\min(n,m)}$ and intersection of (x^n) and (x^m) is $x^{\max(n,m)}$. Hence, it suffices to check that

$$\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c)), \min(a, \max(b, c)) = \max(\min(a, b), \min(a, c)).$$

If a is maximal, then first equality holds. If b is maximal then $\max(a, c) = \min(b, \max(a, c)) = \max(a, c)$. c case is equal. Thus, first one is true for any case. For the second one, check the same for when a (or b or c) is minimal.

9. As hint appeared, it is equivalent to saying that the sequences of A-modules

$$A \xrightarrow{\phi} \bigoplus_{i=1}^{n} A/\mathfrak{a}_i \xrightarrow{\psi} \bigoplus_{i<j} A/(\mathfrak{a}_i + \mathfrak{a}_j)$$

is exact. To see this, if x is solution, then $\phi(x)$ is the solution, thus by exactness $\psi(\phi(x)) = 0$ since each x_i is congruent with x_j mod $\mathfrak{a}_i + \mathfrak{a}_j$. Conversely, if $(x_1, \cdots, x_n)$ satisfies such congruency, then $\psi(x_1, \cdots, x_n) = 0$, thus by exactness, there is a solution $x \in A$ such that $\phi(x) = (x_1, \cdots, x_n)$.

To see it is exact, think about it by localization over $\mathfrak{p}$. By Proposition 3.3,

$$A_{\mathfrak{p}} \xrightarrow{\phi} \bigoplus_{i=1}^{n} (A/\mathfrak{a}_i)_{\mathfrak{p}} \xrightarrow{\psi} \bigoplus_{i<j} (A/(\mathfrak{a}_i + \mathfrak{a}_j))_{\mathfrak{p}}$$

is still exact. And, as we have shown in the proof of 7, $(A/\mathfrak{a}_i)_{\mathfrak{p}} \cong A_{\mathfrak{p}}/\mathfrak{a}_i A_{\mathfrak{p}}$ and $(A/(\mathfrak{a}_i + \mathfrak{a}_j))_{\mathfrak{p}} \cong A_{\mathfrak{p}}/(\mathfrak{a}_i + \mathfrak{a}_j)A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is DVR, $\mathfrak{a}_i A_{\mathfrak{p}} \cong (x^{n_i})$ where x is an element generates $\mathfrak{p}A_{\mathfrak{p}}$. Now, assume that $n_i < n_j$ if $i < j$. Then, $(\mathfrak{a}_i + \mathfrak{a}_j)A_{\mathfrak{p}} = (x^{n_i})$ if $i < j$.

Thus, for any $(x_1, \cdots, x_n) \in \bigoplus_{i=1}^{n} (A/\mathfrak{a}_i)_{\mathfrak{p}}$, each (i, j) component of $\psi(x_1, \cdots, x_n))$ is $x_i - x_j + \mathfrak{a}_i + \mathfrak{a}_j$. If $(x_1, \cdots, x_n) \in \ker \psi$, then $x_i - x_j \in \mathfrak{a}_i + \mathfrak{a}_j = \mathfrak{a}_i$ when $i < j$. Thus, $x_1 - x_n \in \mathfrak{a}_1, x_2 - x_n \in \mathfrak{a}_2, \cdots, x_{n-1} - x_n \in \mathfrak{a}_{n-1}$. This implies that $\phi(x_n) = (x_1, \cdots, x_n)$. Hence, $\ker \psi \subseteq \operatorname{Im} \phi$. Conversely, $\psi \circ \phi(x) = 0$ since $\phi(x) = (x, \cdots, x)$ and (i, j)-component of $\psi \circ \phi(x)$ is $x - x + \mathfrak{a}_i + \mathfrak{a}_j = 0$.

Hence the sequence is exact locally. By Proposition 3.9, applying each map, we have the exact sequence on the original one.

Notes that exactness is also a local property.

10 Completions

Notes that $(x, y) \mapsto xy$ is continuous, then $T_a : G \to G$ by $(a, y) \mapsto ay$ is continuous for any $a \in G$, since T_a is a restriction of continuous map (from $\{a\} \times G$ to G), thus continuous. Also, $f : y \mapsto -y$ is continuous, so does $2f$. Hence, $(x, y) \mapsto x + y \mapsto x + y - 2y = x - y$ is continuous.

And, T_a and T_{-a} are bijectively continuous, and open map since $T_a(O) = O+a$ and since $T_{-a}(O+a) = O$, $T_{-a}^{-1}(O) = O + a$, thus $O + a$ is open, thus T_a is an open map. Hence, it is homeomorphism.

Lemma 10.1. *Let H be the intersection of all neighborhoods of 0 in G. Then,*

1. *H is a subgroup*
2. *H is the closure of $\{0\}$.*
3. *G/H is Hausdorff.*
4. *G is Hausdorff $\iff$ $H = 0$.*

Proof. Let $x \in H$, U be an arbitrary open neighborhood of 0. Then, $-U$ is also open since $x \mapsto -x$ is continuous map. Thus, $V = U \cap -U$ is also an open neighborhood of 0. Thus, from $x \in V$, $-x \in V$. This implies $-x \in U$. Since U was arbitrarily chosen, $-x \in H$. Thus it is closed under inversion.

Also, if $x, y \in H$, then for any U, let $W = +^{-1}(U)$. Then, W contains 0, thus it is open neighborhood of 0, thus $x, y \in W$, which implies $x + y \in U$. Since U was arbitrarily chosen, $x + y \in H$.

Lastly, $0 \in H$.

For 2), $x \in H$ iff for any open neighborhood U of 0, since $x \in U$, $x - U$ contains 0. This is iff all open neighborhoods of x contains 0. This implies x is boundary point of $\{0\}$, thus $x \in \overline{\{0\}}$.

For 3), notes that closed set of quotient topology is union of sets of equivalence classes whose union of those equivalence classes are closed. Since each equivalence class is closed in G, thus each singleton in G/H is closed. Notes that G/H is still an topological group, thus $(x, y) \mapsto x - y$ is continuous, thus preimage of $[0]$ in G/H by this map is closed. Since preimage is just $([x], [x])$, this implies that diagonal in $G/H \times G/H$ is closed. Then by the proposition in topology that diagonal of $X \times X$ is closed iff X is Hausdorff, G/H is Hausdorff.

If G is Hausdorff, then any singleton is closed, thus $H = 0$. If $H = 0$, then $G/H \cong G$, thus G is Hausdorff by iii). ☐

Claim LIII. *X is Hausdorff iff diagonal D of $X \times X$ is closed.*

Proof. If X is Hausdorff, then for any $(x, y) \notin D$, $x \neq y$ thus $\exists U, V$ disjoint open neighborhood of x and y. Thus, $U \times V$ contain (x, y) but disjoint with D, otherwise $U \cap V \neq \emptyset$, contradiction. Hence, $X \times X - D$ is union of such open neighborhoods for all elements of $X \times X - D$, so open. This implies D is closed.

Conversely, if D is closed, take an open neighborhood of $(x, y) \in X \times X - D$. Since $X \times X$ is product topology, we may assume that there is open neighborhood generated by open set in generator of box topology (since $X \times X$ is finite proudct, box topology is equal to product topology), say $U \times V$. Since $U \times V \subseteq X \times X - D$, $U \times V \cap D = \emptyset$. This implies U, V are disjoint open set in X containing x and y respectively. So X is Hausdorff. ☐

Definition Definition of Convergence. *$x_v \to y$ means that for any open neighborhood U of y, $\exists N \in \mathbb{N}$ such that $\forall v > N$, $x_v \in U$.*

Lemma . *Cauchy equivalence is equaivalence relation. And sum of Cauchy sequence is Cauchy. Also, image of Cauchy under continuous homomorphism, say f, is Cauchy.*

Proof. $x_v \sim x_v$ is trivial. If $x_v \sim y_v$, then take any open set U. Let $V = U \cap (-U)$. (V is nonempty since $0 \in V$) Thus, $\exists N \in \mathbb{N}$ such that $x_v - y_v \in V$ for all $v > N$. This implies $y_v - x_v \in V$ since V consists of elements whose inverse is also in V (To see this, if $x \in V$, then $x \in U$ and $-U$, thus $-x \in U$ and $-U$, which implies $-x \in V$.) Thus, $y_v - x_v \in U$ for all $v > N$. Hence $y_v - x_v \to 0$.

To see transitivity, let $x_v - y_v \to 0, y_v - z_v \to 0$. Let U be an open neighborhood of 0. Then, $+^{-1}(U) \subseteq G \times G$ is open set since $+$ is continuous function. Now take $U_1 \times U_2$, a basic open set generated from product topology such that U_1, U_2 are open and $U_1 \times U_2 \subseteq +^{-1}(U)$. Then $U_1 + U_2 \subseteq U$. Take $V = U_1 \cap U_2$. Then, $\exists N \in \mathbb{N}$ such that both $x_v - y_v$ and $y_v - z_v$ are in V when $v > N$. Thus, there sum, $x_v - z_v$ is in $V + V \subseteq U_1 + U_2 \subseteq U$, whenever $v > N$.

Now for sum of two Cauchy sequence, for any open set U, by the same technique we used above, $\exists V \subseteq U$ such that $V + V \subseteq U$. Now take N such that both cauchy difference lies in V. Then, there sum lies in U whenever $v > N$.

For $f(x_v)$, any open set U in the codomain, take preimage and get N for x_v. This N works for $f(x_v)$. ☐

Notes that $\hat{G}$ has topology; for each open set U in G, define

$$\hat{U} := \{\hat{x} \in \hat{G} : \forall \{x_i\}_{i \in \mathbb{N}} \in \hat{x}, \exists N \in \mathbb{N} \text{ s.t. } x_j \in U \text{ for all } j > N\}.$$

These are base of topology $\hat{G}$.

Lemma L. *et $f : G \to H$ continuous homomorphism. Then induced map $\hat{f} : \hat{G} \to \hat{H}$ is continuous homomorphism. Also, $\phi : G \to \hat{G}$ is continuous. Moreover, $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.*

Proof. It is homomorphism since f induces additive homomorphism. Let $\hat{U}$ be an open set in $\hat{H}$. Then, think about $\hat{f}^{-1}(\hat{U})$. If $\{x_v\} \in \hat{f}^{-1}(\hat{U})$, then $\widehat{f(x_v)} \in \hat{U}$, which implies that $f(x_v)$ is eventually in U. Thus, x_v is eventually in $f^{-1}(U)$. Thus, $\hat{f}^{-1}(\hat{U}) \subseteq \widehat{f^{-1}(U)}$. Conversely, if $x_v \in \widehat{f^{-1}(U)}$, the x_v lies eventually in $f^{-1}(U)$, thus $f(x_v)$ lies eventually in U, thus $f(x_v) \in \hat{U}$, thus $x_v \in \hat{f}^{-1}(\hat{U})$. This implies $\hat{f}^{-1}(\hat{U}) = \widehat{f^{-1}(U)}$.

For the second, $\phi^{-1}(\hat{U})$ is a constant whose constant Cauchy sequence eventually lies in U, thus the contstant itself lies in U, hence, $\phi^{-1}(\hat{U}) = U$. Hence ϕ is continuous.

Let x_v be a Cauchy. Then, $\widehat{g \circ f}(x_v) = (g \circ f(x_[v])$ and $\hat{g} \circ \hat{f}(x_v) = \hat{g}(f(x_v)) = g \circ f(x_v)$. Done. □

Now, in the last paragraph of [3][p.102], the author introduce the Fundamental system of neighborhood and a topology generated by the system.

Definition [Fundamental System of neighborhood, or a neighborhood basis]. $0 \in G$ *has a fundamental system of neighborhoods when there is a sequence of neighborhoods of* 0

$$G = G_0 \supseteq G_1 \supseteq \cdots$$

such that for every neighborhood U *of* 0 *in* G $\exists n \in \mathbb{N}$ *such that* $U \supseteq G_n$.

If every G_i *is a subgroup of* G*, then we say it is a fundamental system of neighborhoods consisting of subgroups.*

Recall that neighborhood of 0 means a subset V of A having an open set U such that $0 \in U \subseteq V$. I think he implicitly assume that for any $g \in G$, g has a fundamental system of neighborhood $g + G \supseteq g + G_1 \supseteq \cdots$. Now let $B(x)$ be a fundamental system of neighborhood.

Lemma F. *or any* $g \in G$, $g + G_0 \supseteq g + G_1 \supseteq \cdots$ *forms a neighborhood basis.*

Proof. If U is neighborhood of g, then $-g + U$ is neighborhood of 0. Hence it contains G_n for some n. Thus, U contains $g + G_n$. □

Now we will define a topology from these fundamental system of neighborhoods. Actually, it will be the topology whose basis is a set of all cosets $g + G_n$ for all $g \in G, n \in \mathbb{N}$. In this kind of construction, Atiyah's argument showing G_n is closed is a little bit tautological; if we don't know topology, then we cannot say that whether translation is homeomorphism or not.

Lemma I. *n the topologies, the subgroup* G_n *of* G *are both open and closed.*

Proof. Since $g + G_n$ is in the basis of topology for all $g \in G$, G_n is open. Conversely, $G_n^c = \bigcup_{h \notin G_n} (h + G_n)$. Now, since we know $h + G_n$ is open as a element of basis, G_n is also closed. □

Typical example is the p-adic topology on $\mathbb{Z}$ with $G_n = p^n\mathbb{Z}$.

Lemma T. *he group* G *with this topology is a topological group. Also, translation is continuous map.*

Proof. We use the neighborhood definition of continuity. From $0 + x = x$, we can denote $x + y$ be an arbitrary element of G for some $x, y \in G$. Then, for any open neighborhood $x + y + G_n$, we have a product of open neighborhood $(x + G_n) \oplus (y + G_n)$ such that $(x + G_n + y + G_n) \subseteq x + y + G_n$. So, $\mu(x, y) = x + y$ is continuous.

Similarly, for any open neighborhood $x + G_n$, there is an open neighborhood $-x + G_n$ such that $-(-x + G_n) = x - G_n = x + G_n$ since $-G_n = G_n$ because G_n is a subgroup. Hence, $x \mapsto -x$ is continuous map.

Lastly, let T_a be a translation map. Then, for any $b + G_n$, $\exists b - a + G_n$ such that $T_a(b - a + G_n) = a + b - a + G_n \subseteq b + G_n$. So translation is continuous. □

In the proof of Proposition 10.2, commutativity of the diagram comes from commutativity of the exact sequence of inverse system.

In the proof of Corollary 10.3, $G' \to G \to G/G_n \to 0$ has a kernel $G' \cap G_n$. Thus, $\iota : G'/G' \cap G_n \to G/G_n$ is injective. Also, from $G \to G'' \to G''/p(G_n)$ is surjective and G_n is in the kernel of this map, the map $\pi : G/G_n \to G''/p(G_n)$ is surjective. And $\pi \circ \iota = 0$ by construction; both use the maps in exact sequence. Conversely, if $\overline{g} \in G/G_n$ is in kernel of π, then $\pi(\overline{g}) = \overline{p(g)}$ thus $p(g) \in p(G_n)$. This implies $\exists h \in G_n$ such that $p(g - h) = 0$. By exactness, $g - h \in G'$ by identifying G' as a subgroup of G. Thus, $\iota(g - h + G' \cap G_n) = g - h + G_n = g + G_n = \overline{g}$, thus $\overline{g} \in \operatorname{Im} \iota$. Hence the given sequence in p.105 is exact.

Since $G'/G' \cap G_n$ is surjective system, so by Proposition 10.2, there inverse limits also form an exact sequence.

For Corollary 10.4, $\widehat{G/G_n} = \varprojlim_m (G/G_n)/(G_m/G_n)$. Thus, if $m \geq n$,then $(G/G_n)/(G_m/G_n) \cong G/G_m$ but $m < n$ then $(G/G_n)/(G_m/G_n) \cong G/G_n$. Therefore if (a_n) is a coherent sequence of $\widehat{G/G_n}$, then it has $a_1 = a_2 = \cdots = a_n$. Thus, we have a map $\phi : \widehat{G/G_n} \to G/G_n$ by sending (a_n) to a_1. This map is clearly surjection. To show it is injective, notes that if two coherent sequence $(a_n), (b_n)$ are the same, then $b_2 = g_1 + a_2$ for some $g_1 \in G_1$ and $b_3 = g_2 + a_3$ and so on. Thus $0, g_1, g_2, \cdots$ form a coherent sequence. Since each coherent sequence comes from the Cauchy sequence, say (x_n) be a Cauchy sequence inducing $0, g_1, g_2, \cdots$. Then, limit of x_n in G is 0. Thus, limit of x_n in G/G_k is also zero for all p since map $G \to G/G_n$ is continuous map. (Both are topological spaces;) Thus, $g_1 = g_2 = \cdots = 0$. Thus $(a_n) = (b_n)$ iff $a_n = b_n$ in each G/G_n. Hence, $\phi((a_n)) = 0$ implies $a_1 = 0$, thus $a_1 = \cdots = a_n = 0$, and this 0 sequence is uniquely determined by Cauchy sequence inducing (a_n), thus $(a_n) = (0)$. This shows injectivity of ϕ. Also it is clear that ϕ is additive homomorphism. Thus ϕ is isomorphism.

Now we claim that $\widehat{G/G_n} \cong \widehat{G}/\widehat{G_n}$. To see this, take a map $\phi : \widehat{G/G_n} \to \widehat{G}/\widehat{G_n}$ by $(a_n) \mapsto \overline{(b_n)}$ where $b_m = a_m$ for $m > n$ and $b_i = \theta_{i+1} b_{i+1}$ for $m \leq n$. Then, $a_n = b_n$, and this map is well-defined since (a_n) has unique representation. Moreover, it is injective, since if $\overline{(b_i)} = 0$ then $(b_i) \in \widehat{G_n}$, thus $b_1 = \cdots = b_n = 0$ since limit of a Cauchy sequence inducing a_n lies in G_n. Hence, $(a_n) = 0$ in G/G_n which is isomorphic to $\widehat{G/G_n}$. Also, ϕ is surjective since any sequence $(b_1, b_2, \cdots) \in \widehat{G}$ can be represented taking $a_n = b_n$ in $\widehat{G/G_n}$ with the map of inverse system θ. Hence, ϕ is isomorphism.

And $\mathfrak{a}$-adic topology is a topology having $\{\mathfrak{a}^i\}_{i \in \mathbb{N}}$ as a fundamental system of neighborhood of 0. So the basis of this topology is a collection of all cosets $g + \mathfrak{a}^i$ for any g.

Claim LIV. *$\mathfrak{a}$-adic topology on A makes A a topological ring.*

Proof. We already showed that addition and (additive) inversion are continuous. So it suffices to show that $\mu_\times(x, y) = xy$ is continuous. Recall that the neighborhood definition of continuity. Also notes that $x = 1 \cdot x$ in A, so any element in A can be denoted as xy for some $x, y \in A$. Then, for any neighborhood of xy, say $xy + \mathfrak{a}^n$, there is an open neighborhood of (x, y), which is $(x + \mathfrak{a}^n) \times (y + \mathfrak{a}^n)$ such that $(x + \mathfrak{a}^n) \cdot (y + \mathfrak{a}^n) = xy + x\mathfrak{a}^n + y\mathfrak{a}^n + \mathfrak{a}^{2n} \subseteq xy + \mathfrak{a}^n$, since for any $m_1 \cdots m_n \in \mathfrak{a}^n$ with $m_i \in \mathfrak{a}$, $xm_1 \cdots m_n = (xm_1) \cdots m_n \in \mathfrak{a}^n$. So multiplication is continuous. ☐

In the proof of Lemma 10.8, Q_n is finitely generated as an A-module implies M_n^* is finitely generated as A^* module, since $\mathfrak{a}^k$ part acting on M_n components yields $\mathfrak{a}^k M_n$, thus no more generators are needed to construct M_n^*.

In the proof of Proposition 10.9, actually the author uses Lemma 10.8 twice; He use it for showing that M_n is a stable filtraion implies M^* is a finitely generated module, and use it again showing that the finitely generated A^* module $(M' \cap M)^*$ gives a stable $\mathfrak{a}$-filtration.

In the proof of Corollary 10.10, notes that $\mathfrak{a}^n M$ is already $\mathfrak{a}$-stable filtration by definition. In the proof of Theorem 10.11, observe that $\mathfrak{a}^n M'$ is stable filtration, thus by Lemma 10.6, we have a bounded difference. Coinciding topology comes from the fact that in a bounded difference we can always find an open neighborhood of one topology which is contained in an open neighborhood of the other topology.

For the proof of Proposition 10.12, we need to assure that the sequence $M' \cap (\mathfrak{a}M)$ and $p(\mathfrak{a}M)$ gives the same topology with the completion by $\mathfrak{a}M'$ and $\mathfrak{a}M''$. This is assured by Lemma 10.6; notes that $M' \cap (\mathfrak{a}M)$ is stable by Artin-Rees Lemma, and $p(\mathfrak{a}M)$ is also stable since p is an A-homomorphism so $\mathfrak{a}(\mathfrak{a}^n p(M)) = p(\mathfrak{a}^n M) = p(\mathfrak{a}^{n+1} M) \subseteq \mathfrak{a}^{n+1} p(M)$. Hence, by Lemma 10.6, all stable filtration induce the same topology.

In [3][p.108], the third map is precisely

$$\hat{A} \bigotimes_A M \xrightarrow{1_{\hat{A}} \otimes f} \hat{A} \bigotimes_A \hat{M} \xrightarrow{\delta} \hat{A} \bigotimes_{\hat{A}} \hat{M} = \hat{M},$$

where $f : M \to \hat{M}$ is canonical injection by construction of $\hat{M}$. And δ is defined by identity map. Also notes that δ is well-defined via bilinear map $A \times M \to A \times M$ corresponding to δ, which is identity.

Also, Corollary 10.3 induces that completion commutes with finite direct sum by thinking direct sum as a split exact sequence.

In the proof of Proposition 10.13, δ is surjective since $0 \to \ker(\hat{A} \bigotimes_A F \to \hat{A} \bigotimes_A M) \to \hat{A} \bigotimes_A F \to \hat{A} \bigotimes_A M \to 0$ is exact and apply Corollary 10.3. This implies α is surjective; for any $x \in \hat{M}$, β is isomorphism and δ is surjective means $\exists y \in \hat{A} \bigotimes_A F$ such that $\delta \circ \beta(y) = x$. This implies $\alpha \circ (\hat{A} \bigotimes_A F \to \hat{A} \bigotimes_A M)(y) = x$, hence α is surjective. If we assume Noetherian, since N is a submodule of finitely generated module, thus it is finitely generated. Hence by the argument what we did now, γ is surjective. And we know bottom line is exact by applying 10.12 on $0 \to N \to F \to M \to 0$. Now, if $x \in \ker(\alpha)$, then $\exists y \in \hat{A} \bigotimes_A F$ such that its image is x, thus $\delta \circ \beta(x) = 0$. Hence, $\beta(x)$ is in the kernel of of δ, thus it is image of $z \in \hat{N}$. By surjectivity of γ, z is image of $z' \in \hat{A} \bigotimes_A N$. By commutativity of diagram, sending z' along the top line, we have x is image of z'. Since the top line is exact, $x = 0$. So α is injective.

Proposition 10.14 is direct application of definition of flat.

In the proof of Proposition 10.15, actually, we cannot directly apply 10.4 since $\hat{A}$ is completion of ring, not a group. However, by the same argument we did for 10.4, we get the same statement for the completion of ring. And for iv), let $\xi_n = \sum_{j=0}^{n} x^j = 1 + x + \cdots + x^n$. Then, ξ_n is Cauchy in $\hat{A}$, since for any $\hat{\mathfrak{a}}^k$, there is $N = k$ such that $\forall n > m > N$ $\xi_n - \xi_m = x^{m+1} + \cdots x^n \in \hat{\mathfrak{a}}^k$ since $n > \cdots > m+1 > k$. Thus, limit of ξ_n converges, since $\hat{A}$ is complete. From this, $(1-x)$ is unit, since $(1-x) \cdot \lim \xi_n = 1$. Since it holds for all $x \in \hat{\mathfrak{a}}$, thus for any $y \in \hat{A}$, $1 - xy$ is unit since $xy \in \hat{\mathfrak{a}}$. This implies x is element of Jacobson radical by Proposition 1.9. Since x was arbitrarily chosen, $\hat{\mathfrak{a}}$ is contained in the Jacobson radical.

In the proof of Theorem 10.17, known as Krull's intersection theorem, $\mathfrak{a}E = E$ follows from Artin-Rees lemma; let k be the k in Corolllary 10.10. Let $M' = \bigcap_{n \neq k}^{\infty} \mathfrak{a}^n M$. Then, $E = \mathfrak{a}^k M \cap M'$. By Artin-Rees lemma,

$$\mathfrak{a}E = \mathfrak{a}(\mathfrak{a}^k M \cap M') \underbrace{=}_{\text{A-R Lemma}} (\mathfrak{a}^{k+1}M) \cap M' = (\mathfrak{a}^{k+1}M) \cap (\bigcap_{n \neq k}^{\infty} \mathfrak{a}^n M) = (\bigcap_{n \neq k}^{\infty} \mathfrak{a}^n M) = M'.$$

However, notes that $\mathfrak{a}^n M \supseteq \mathfrak{a}^m M$ if $m > n$. Thus, $M' = E$ since $M' = \bigcap_{n>k}^{\infty} \mathfrak{a}^n M = E$.

In the definition of $G(A)$, if $\overline{x_m} = \overline{x_m + x_{m+1}}$ and $\overline{x_n} = \overline{x_n + x_{n+1}}$, then $(x_m + x_{m+1}) \cdot (x_n + x_{n+1}) = x_m x_n + x_m x_{n+1} + x_{m+1} x_n + x_{n+1} x_{m+1} \equiv x_m x_n \mod \mathfrak{a}^{m+n+1}$. Thus multiplication of $G(A)$ is well-defined.

To see $G(A) = (A/\mathfrak{a})[\overline{x_1}, \cdots, \overline{x_s}]$ in p.111 of [3], notes that set of monomials of degree k is generating set of $\mathfrak{a}^k/\mathfrak{a}^{k+1}$ by definition of multiplication in $G(A)$. Also, it has elements in $A/\mathfrak{a}$. And if $\overline{y_0} \in A/\mathfrak{a}$, then $\overline{y_0} \cdot \overline{x_l} = \overline{y_0 x_l}$. Since $y_0 \notin \mathfrak{a}$ if $\overline{y_0} \neq 0$, so $y_0 x_l \notin \mathfrak{a}^2$. Hence, $\overline{y_0 x_l}$ is nonzero. Thus, this kind of identification shows $G(A) = (A/\mathfrak{a})[\overline{x_1}, \cdots, \overline{x_s}]$ as a set. Thus the identification is bijection, and additive homomorphism in clear way. Also, the identification of case $\overline{y_0 x_l}$ gives multiplicative homomorphism as well. Hence, $G(A) = (A/\mathfrak{a})[\overline{x_1}, \cdots, \overline{x_s}]$ as a ring.

For the proof of Lemma 10.23, actually we need induction to show that either $\ker \alpha_n = 0$ or $\operatorname{coker} \alpha_n = 0$. To see this, when $n = 0$, then $\alpha_n : 0 \to 0$ thus both injective or surjective. Then, do the diagram chasing to show that α_{n+1} is either injective or surjective, when α_n is either injective or surjective, depends on whether $G_n(\phi)$ is injective or surjective. (Actually, it is very similar to intermediate step of Five lemma.) In case of $G_n(\phi)$ is injective, then α_n is injective, thus by taking inverse limit on the exact sequence derived by α_n we have an injective map from $\varprojlim(A_n) \to \varprojlim(B_n)$ since the functor of inverse limit is left exact, by Proposition 10.2. In case of $G_n(\phi)$ is surjective, then α_n is surjective. Also, we know from the proof of Proposition 10.12 that since $G_n(\phi)$ gives surjective system on $G(A)$, which implies that taking inverse limit gives surjective map between two inverse limit.

In the proof of Proposition 10.24, to see that β is injective, let $x \in \ker \beta$. Then, the constant Cauchy sequence $(x)_{n\in\mathbb{N}}$ is zero in M/M_n for all n, which implies $x \in \bigcap_{n\in\mathbb{N}} M_n = \emptyset$ since M is Hausdorff in its filtration topology. Thus, $x = 0$. Also, to see that ϕ is surjective, let $m \in M$ such that $m \neq 0$. Then, $\beta(m) \neq 0$, thus by the surjectivity of $\hat{\phi}$ and α, $\exists f \in F$ such that $\hat{\phi}(\alpha(f)) = \beta(m)$. Thus, $\beta \circ \phi(f) = \beta(m)$. Since β is injective, $\phi(f) = m$. Hence ϕ is surjective. Thus, M is generated by $x_1, \cdots, x_r$.

Now, for Exercise 1, define

Definition T. *he concept p-adic completion is defined as $\varprojlim_{n\geq 1} A/(p^n A)$.*

1. By definition, $\hat{A} = \varprojlim_{k\geq 0} A/(p^k A)$, where $A = \bigoplus_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}$. Thus, $p^k A = 0$ for any $k > 1$, which implies that $\hat{A} = \varprojlim_{k\geq 0} A/(p^k A) = \bigoplus_{k\geq 0} A/(0) = A$. Also, notes that $B = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$. Thus $p^k B = \bigoplus_{n\in\mathbb{N}} p^k\mathbb{Z}/p^n\mathbb{Z} \cong \bigoplus_{n=k+1}^{\infty} p^k\mathbb{Z}/p^n\mathbb{Z}$. Thus, from

$$\alpha_n^{-1}(p^k\mathbb{Z}/p^n\mathbb{Z}) = \begin{cases} 0 & \text{if } k \leq n \\ \{x \in \mathbb{Z}/p\mathbb{Z} : p^{n-1}x \in p^k\mathbb{Z}/p^n\mathbb{Z}\} = \mathbb{Z}/p\mathbb{Z} & \text{if } k > n \end{cases}$$

we know that $\alpha^{-1}(p^k B) \cong \bigoplus_{n>k} \mathbb{Z}/p\mathbb{Z}$. Hence, by letting $A_k = \alpha^{-1}(p^k B) \cong \bigoplus_{n>k} \mathbb{Z}/p\mathbb{Z}$, $A/A_k = (\mathbb{Z}/p\mathbb{Z})^k$. Thus, it suffices to show that $\varprojlim(\mathbb{Z}/p\mathbb{Z})^k = \prod_{n\geq 0} \mathbb{Z}/p\mathbb{Z}$. Actually, it is just special case of more general one;

Claim LV. *Let $A = \bigoplus_{n\in\mathbb{N}} R_n$ and $A_k = \bigoplus_{n>k} R_n$. Then, $\varprojlim_{n\in\mathbb{N}}(A/A_n) = \varprojlim_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} R_i = \prod_{n\geq 0} R_n$.*

Proof. Notes that $\theta : A/A_{n+1} \to A/A_n$ is just canonical projection, by deleting last position. Let (a_n) be a coherent sequence in A. Then each a_n is n-dimensional tuple such that $\theta_{n+1}(a_{n+1}) = a_n$. Thus, first n position of a_{n+1} is equal to a_n.

Now define a map $\phi : \prod_{n\in\mathbb{N}} R_n \to \varprojlim_{n\in\mathbb{N}}(A/A_n)$ by $(r_1, \cdots,) \mapsto (a_n)$ where $a_n = (r_1, \cdots, r_n)$. Then first of all, it is well-defined since (a_n) form a coherent sequence. Conversely, let $\psi : \varprojlim_{n\in\mathbb{N}}(A/A_n) \to \prod_{n\in\mathbb{N}} R_n$ by $(a_n) \mapsto (r_1, \cdots,)$ where r_n is the last component of a_n. Then, $\phi\circ\psi$ and $\psi\circ\phi$ is inverse to each other. Also, they are ring homomorphism by checking additive and multiplicative homomorphism. (It is tedious but trivial, so omit it.) Hence, ϕ and ψ are ring isomorphism. $\square$

To see that p-adic completion is not a right exact, observe that

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{p\cdot - : x \mapsto p\cdot x} B \to 0$$

is exact, since kernel of $x \mapsto p \cdot x$ is $\bigoplus_{n\geq 0} p^{n-1}\mathbb{Z}/p^n\mathbb{Z} = \alpha(A)$. Then by p-adic completion we have a map

$$\hat{A} \xrightarrow{\hat{\alpha}} \hat{B} \xrightarrow{\widehat{p\cdot -}} \hat{B} \to 0$$

Notes that this is not exact; to see this, notes that $\hat{B} \cong \prod_{n\geq 0} \mathbb{Z}/p^n\mathbb{Z}$. To see this, notes that $B/B_k = B/p^k B \cong (\bigoplus_{n=1}^{k} \mathbb{Z}/p^n\mathbb{Z}) \bigoplus (\bigoplus_{n\geq k} \mathbb{Z}/p^k\mathbb{Z})$. Hence, if (b_n) is a coherent sequence in $\varprojlim B/B_k$, then first n components of b_{n+1} is equal to first n component of b_n. Hence, take a map $\phi : \hat{B} \to \prod_{n\geq 0} \mathbb{Z}/p^n\mathbb{Z}$ by $(b_n) \mapsto (r_1, \cdots)$ where r_n is n-th component of b_n. Conversely, take a map $\psi : \prod_{n\geq 0} \mathbb{Z}/p^n\mathbb{Z} \to \hat{B}$ by taking canonical projection for each component in $(r_1, \cdots)$. Then, it is well-defined inverse of each other, and you can see that it satisfies homomorphic property.

From this observation, $\widehat{p\cdot -}(r_1, \cdots) = 0$ implies $r_i \in p^{i-1}\mathbb{Z}/p^i\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$ as a subset of $\mathbb{Z}/p^i\mathbb{Z}$. Thus, $\ker(\widehat{p\cdot -}) = \prod_{n\geq 0} p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$. However, notes that $\alpha(A) = \bigoplus_{n\geq 0} p^{n-1}\mathbb{Z}/p^n\mathbb{Z} \subsetneq \prod_{n\geq 0} p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$, thus the given sequence is not exact.

2. By above computation, we know that the give exact sequence is

$$0 \to \bigoplus_{k>n} \mathbb{Z}/p\mathbb{Z} \to \bigoplus_{n\geq 0} \mathbb{Z}/p\mathbb{Z} \to \bigoplus_{n=0}^{k} \mathbb{Z}/p\mathbb{Z} \to 0.$$

Thus, $\varprojlim_{n\geq 0} A/A_n = \prod_{n\geq 0} \mathbb{Z}/p\mathbb{Z}$ as we've shown above. Also, to calculate $\varprojlim_{n\geq 0} A_n$, notes that an inverse system commute with A/A_n is $\theta_{n+1} : A_{n+1} \to A_n$ by injection; this means that if $a_{n+1} = (a_{n+1,1}, \cdots) \in A_{n+1}$, then $\theta(a_{n+1}) = (0, a_{n+1,2}, \cdots)$ in A_n. Thus, if we let (a_n) is a coherent sequence in $\varprojlim_{n\geq 0} A_n$, then n-th component of a_1 comes from the first component of a_n. However, from $a_n = \theta(a_{n+1})$, so the first component of a_n is zero. This implies n-th component of a_1 is zero. Since

n was arbitrarily chosen, $a_1 = (0, \cdots)$. By the same argument on induction, we have $a_n = (0, \cdots)$ for all n. Thus, $\varprojlim_{n \geq 0} A_n = 0$. Thus, what we have by taking $\varprojlim$ functor is

$$0 \to 0 \to A = \bigoplus_{n \geq 0} \mathbb{Z}/p\mathbb{Z} \to \varprojlim_{n \geq 0} A/A_n = \prod_{n \geq 0} \mathbb{Z}/p\mathbb{Z} \to 0$$

Since $\bigoplus_{n \geq 0} \mathbb{Z}/p\mathbb{Z} \neq \prod_{n \geq 0} \mathbb{Z}/p\mathbb{Z}$, it is not right exact. (Notes that $\varprojlim A = A$ since every coherent sequence (a_n) should satisfy $a_n = a_{n+1}$ since θ is identity map.)

Now to calculate $\varprojlim^1 A_n = \operatorname{coker}(d^A)$ where $d^A : \prod_{n=1}^{\infty} A_n \to \prod_{n=1}^{\infty} A_n$ by $a_n \mapsto a_n - \theta_{n+1}(a_{n+1})$, we can use the Snake lemma in the proof of Proposition 10.2, saying that

$$0 \to \varprojlim_{n \geq 0} A_n \to \varprojlim_{n \geq 0} A \to \varprojlim_{n \geq 0} A/A_n \to \varprojlim^1 A_n \to \varprojlim^1 A \to \varprojlim^1 A/A_n \to 0.$$

From above calculation, we have

$$0 \to 0 \to \bigoplus_{n \geq 0} \mathbb{Z}/p\mathbb{Z} \to \prod_{n \geq 0} \mathbb{Z}/p\mathbb{Z} \to \varprojlim^1 A_n \to \varprojlim^1 A \to \varprojlim^1 A/A_n \to 0.$$

Now we claim that $\varprojlim^1 A = 0$. To see this, let $B_n = A, B = \prod_{n \in \mathbb{N}} B_n$ for distinguishing notation. Then, $d^B : B \to B$ is by $d^B(a_n) = a_n - \theta_{n+1}(a_{n+1})$. However, since θ is identity map, $d^B(a_n) = a_n - a_{n+1}$. Thus, for any $(a_1, \cdots) \in B$, we have an element $(0, -a_1, \cdots)$ such that $d^B(0, -a_1, -a_1 - a_2, \cdots) = (0 - (-a_1), -a_1 - (-a_1 - a_2), \cdots) = (a_1, a_2, \cdots)$. This implies that d^B is surjective. Hence, $\operatorname{coker}(d^B) = 0$, which implies $\varprojlim^1 A = 0$. Therefore, we have an exact sequence

$$0 \to \bigoplus_{n \geq 0} \mathbb{Z}/p\mathbb{Z} \to \prod_{n \geq 0} \mathbb{Z}/p\mathbb{Z} \to \varprojlim^1 A_n \to 0 \to \varprojlim^1 A/A_n \to 0.$$

Thus, by the first isomorphism theorem, $\varprojlim^1 A_n \cong \prod_{n \geq 0} \mathbb{Z}/p\mathbb{Z} / \bigoplus_{n \geq 0} \mathbb{Z}/p\mathbb{Z}$.

(It is a little bit weird, since it seems that there is no way to directly show that $\operatorname{coker}(d^A) \cong \bigoplus_{n \geq 0} \mathbb{Z}/p\mathbb{Z}$.)

3. By the Krull's intersection theorem, the left hand side (say E) is all elements in M annihilated by $1+a$ for some $a \in \mathfrak{a}$. If $\mathfrak{m}$ contains $\mathfrak{a}$, then $1+a \in A - \mathfrak{m}$, otherwise $(1+a) - a = 1 \in \mathfrak{m}$, contradiction. Thus, $E_\mathfrak{m} = 0$, since for given $x \in E$ which is annihilated by $1+a$ for some $a \in \mathfrak{a}$, $x/1 = x(1+a)/(1+a) = 0/1$. Hence, $x \in \ker(M \to M_\mathfrak{m})$ if $\mathfrak{m}$ contains $\mathfrak{a}$. Thus, $E \subseteq$ RHS.

   Conversely, let $N = \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \to M_\mathfrak{m})$. Then, $N_\mathfrak{m} = 0$ if $\mathfrak{m} \supseteq \mathfrak{a}$, since $N_\mathfrak{m}$ is submodule of $M_\mathfrak{m}$ and apply the fact that N is inside of kernel. Thus, by Exercise 3.14, $K = \mathfrak{a}K$. Hence, $K = \mathfrak{a}K = \mathfrak{a}^2 K = \cdots$ implies $K = \bigcap_{n \in \mathbb{N}} \mathfrak{a}^n K \subseteq E$.

   For the remaining, notes that $\hat{M} = 0 \iff \hat{M} = \hat{\mathfrak{a}}\hat{M}$, since if part holds by Nakayama lemma with the fact Proposition 10.15 iv) that $\hat{\mathfrak{a}}$ is contained in the Jacobson radical. Also, $\hat{M} = \hat{\mathfrak{a}}\hat{M}$ iff $M = E$, by the argument $\hat{M} = \hat{\mathfrak{a}}\hat{M} = \hat{\mathfrak{a}}^2\hat{M} = \cdots$ as above. And $M = E$ iff $M = K$. Thus, by definition of support, any maximal ideal containing $\mathfrak{a}$ should not be in $\operatorname{Supp}(M)$. Now, notes that if $\mathfrak{p}$ is a prime ideal containing $\mathfrak{a}$ but contained in a maximal ideal $\mathfrak{m}$, then $A - \mathfrak{p}$ contains $1 + a$, thus $E_\mathfrak{p} = 0$, since $1 + a$ is unit in $A_\mathfrak{p}$. Thus, $\mathfrak{p} \notin \operatorname{Supp}(M)$. Since $\mathfrak{p}$ was chosen arbitrarily from a set of prime ideals containing $\mathfrak{a}$, which is $V(\mathfrak{a})$, this implies $\operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset$. (Conversely, if $\operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset$, then $\operatorname{Supp}(M)$ contains no maximal ideal containing $\mathfrak{a}$, trivially.)

4. If x is not a zero divisor, then $0 \to A \xrightarrow{x} A$ is exact. Now, by Proposition 10.14, $\hat{A}$ is flat, thus, $\hat{A} \bigotimes_A -$ is exact functor. This implies $0 \to \hat{A} \xrightarrow{\hat{x}} \hat{A}$ is exact, since $\hat{A} \bigotimes_A A = \hat{A}$.

   The answer for last question is no. This example comes from [10][p.187-188]. Let $R = k[x, y]$ where k is a field of characteristic zero. Then, $\mathfrak{m} := (x, y)$ is the maximal ideal. As we've seen in the proof of

Corollary 10.27, $k[[x,y]]$ is completion of R by $\mathfrak{m}$-adic topology. Now let $A = k[x,y]/(y^2 - x^2 - x^3)$. Then A has maximum ideal $\overline{\mathfrak{m}} = (\overline{x}, \overline{y})$ where $\bar{\cdot}$ is a canonical map from R to A. Then completion with respect to $\overline{\mathfrak{m}}$ is $\hat{A} \cong \hat{R} \otimes RA \cong k[[x,y]]/(y^2 - x^2 - x^3)$. (The last equality can be easily shown by observing that every element is of form $f \otimes \overline{1}$.) Since $y^2 - x^2 - x^3$ is irreducible (if decomposition exists, it has a form $(y + f(x))(y + g(x))$, but this gives $f(x) + g(x) = 0, f(x)g(x) = -x^2 - x^3$, which is nonsense.) so it is prime since R is UFD. Thus, A is integral domain. However, to see $\hat{A}$ is not integral domain, it suffices to show that $y^2 - x^2 - x^3$ is reducible; then from the fact that $k[[x,y]]$ is UFD (since it is power series ring over a field), $y^2 - x^2 - x^3$ not a prime. Thus, $\hat{A}$ is not integral domain.

To see this, notes that $y^2 - x^2 - x^3 = (y + xf)(y - xf) = y^2 - x^2 f^2$ if there exists f such that $f^2 = (1 + x)$. Now, by letting $f = \sum_{n \geq 0} a_n x$, we can inductively define a_n by letting $a_0 = 1$. Thus, such f exists. This is counter example of the question.

5. Since M is finitely generated, by Corollary 10.13,

$$M^{\mathfrak{a}} \cong A^{\mathfrak{a}} \bigotimes_A M, M^{\mathfrak{b}} \cong A^{\mathfrak{b}} \bigotimes_A M.$$

Thus,

$$(M^{\mathfrak{a}})^{\mathfrak{b}} = A^{\mathfrak{b}} \bigotimes_A (A^{\mathfrak{a}} \bigotimes_A M) \underbrace{=}_{\text{Exercise 2.15}} (A^{\mathfrak{b}} \bigotimes_A A^{\mathfrak{a}}) \bigotimes_A M = (A^{\mathfrak{a}})^{\mathfrak{b}} \bigotimes_A M$$

while

$$M^{\mathfrak{a}+\mathfrak{b}} = A^{\mathfrak{a}+\mathfrak{b}} \bigotimes_A M.$$

Thus, it suffices to show that $(A^{\mathfrak{a}})^{\mathfrak{b}} = A^{\mathfrak{a}+\mathfrak{b}}$.

Now, the inclusion induces same topology between $A_n = (\mathfrak{a}^n + \mathfrak{b}^n)A$ and $A'_n = (\mathfrak{a} + \mathfrak{b})^n A$, since for any basic open neighborhood $x + A_n$ contains an open neighborhood $x + A'_{2n}$, thus any open set in the topology induced by A_n is open in another topology induced by A'_n and vice versa. Thus, completion over A_n or A'_n are the same, since completion is also defined by Cauchy sequences, and the same topology implies that the set of equivalence of Cauchy sequence is the same, which means the completion is the same. Thus, from the observation that completion is equal to inverset limit, we have

$$\varprojlim A/(\mathfrak{a}^n + \mathfrak{b}^n)A \cong \hat{A} \cong \varprojlim A/(\mathfrak{a} + \mathfrak{b})^n A.$$

Now, from the given isomorphism, we have

$$\varprojlim A/(\mathfrak{a} + \mathfrak{b})^n A \cong \varprojlim_m (\varprojlim_n A/(\mathfrak{a}^n + \mathfrak{b}^m)A).$$

Now observe that $\varprojlim A/(\mathfrak{a} + \mathfrak{b})^n A \cong A^{\mathfrak{a}+\mathfrak{b}}$ by definition. To see $\varprojlim_m (\varprojlim_n A/(\mathfrak{a}^n + \mathfrak{b}^m)A) \cong (A^{\mathfrak{a}})^{\mathfrak{b}}$, notes that isomorphism $A^{\mathfrak{a}} \bigotimes_A M \to M^{\mathfrak{a}}$ is given by $(a_n) \bigotimes_x \mapsto (a_n x)$ (see p.108 of [3]). Thus, the image of $(\mathfrak{b}^m M)^{\mathfrak{a}} \to M^{\mathfrak{a}}$ is $\mathfrak{b}^m M^{\mathfrak{a}}$. From the exact sequence

$$0 \to \mathfrak{b}^m M \to M \to M/\mathfrak{b}^m M \to 0$$

we have exact sequence by Proposition 10.12

$$0 \to (\mathfrak{b}^m M)^{\mathfrak{a}} \to M^{\mathfrak{a}} \to (M/\mathfrak{b}^m M)^{\mathfrak{a}} \to 0$$

Then by the image of $(\mathfrak{b}^m M)^{\mathfrak{a}} \to M^{\mathfrak{a}}$ with first isomorphism theorem, we have

$$(M/\mathfrak{b}^m M)^{\mathfrak{a}} \cong M^{\mathfrak{a}}/\mathfrak{b}^m M^{\mathfrak{a}}.$$

Thus

$$(A^{\mathfrak{a}})^{\mathfrak{b}} = \varprojlim_m (A^{\mathfrak{a}})/\mathfrak{b}^m A^{\mathfrak{a}} \cong \varprojlim_m (A/\mathfrak{b}^m A)^{\mathfrak{a}} = \varprojlim_m \varprojlim_n (A/\mathfrak{b}^m A)/((\mathfrak{a}^n + \mathfrak{b}^m)/(\mathfrak{b}^m A)) \cong \varprojlim_m \varprojlim_n A/(\mathfrak{a}^n + \mathfrak{b}^m)$$

where before last isomorphism comes from the fact that $(\mathfrak{a}^n + \mathfrak{b}^m)$ is the image of $\mathfrak{a}^n$ under the map $A \to A/\mathfrak{b}^m$,and where the last isomorphism is from the third isomorphism theorem.

Hence, $(A^{\mathfrak{a}})^{\mathfrak{b}} = A^{\mathfrak{a}+\mathfrak{b}}$ holds.

To see that why the given isomorphism in the hint is true, notes that by definition, an element of $\varprojlim_m(\varprojlim_n A/(\mathfrak{a}^n + \mathfrak{b}^m))$ is a coherent sequence $(a_m)_{m\in\mathbb{N}}$ where $a_m \in \varprojlim_n A/(\mathfrak{a}^n + \mathfrak{b}^m)$, i.e., $a_m = (a_{m,n})_{n\in\mathbb{N}}$. Now let $\theta_1 : a_{m+1} \mapsto a_m$ and $\theta_2 : a_{m,n+1} \mapsto a_{m,n}$, which makes the inverse system for each inverse limit. Then, notes that for given $a_{m,n}$, if $n < m$, then iterative application of θ_2 sends $a_{m,m}$ to $a_{m,n}$. If $n > m$, then iterative application of θ_1 sends $a_{n,n}$ to $a_{m,n}$. Thus, the coherent sequence $(a_m)_{m\in\mathbb{N}}$ is deteremined by $(a_{n,n})_{n\in\mathbb{N}} \in \varprojlim_n A/(\mathfrak{a}+\mathfrak{b})^n A$. Also, from given $(b_n) \in \varprojlim_n A/(\mathfrak{a}+\mathfrak{b})^n A$, we can recover $(a_m) \in \varprojlim_m(\varprojlim_n A/(\mathfrak{a}^n+\mathfrak{b}^m))$ by setting $a_{n,n} = b_n$ and define $a_{m,n} = \begin{cases} \theta_1^{m-n} a_{m,n} & \text{if } m > n \\ \theta_2^{n-m} a_{m,n} & \text{if } m < n \end{cases}$. Hence this gives a bijective mapping, moreover preserves additivity and multiplicativity. Thus, it is bijection.

6. If $\mathfrak{a}$ is inside of Jacobson radical, then every maximal ideal $\mathfrak{m}$ contains $\mathfrak{a}$. If $x \notin \mathfrak{m}$, then $(x+\mathfrak{a})\cap\mathfrak{m} = \emptyset$, otherwise $(x+a)-a = x \in \mathfrak{m}$, contradiction. Since $x+\mathfrak{a}$ is open in $\mathfrak{a}$-topology, so the complement of $\mathfrak{m}$ is open, hence $\mathfrak{m}$ is closed.

   Conversely, if $\mathfrak{a}$ is not inside of Jacobson radical, then there exists a maximal ideal $\mathfrak{m}$ not containing $\mathfrak{a}$. Then, for any $n \in \mathbb{N}$, $\mathfrak{a}^n$ is not contained in $\mathfrak{m}$, otherwise its radical $\mathfrak{a}$ should be contained in $\mathfrak{m}$, contradiction. Thus, $\mathfrak{a}^n + \mathfrak{m} = (1)$ for all $n \in \mathbb{N}$. Thus, for given $n \in \mathbb{N}$, we have a sequence $a_n \in \mathfrak{a}^n, m_n \in \mathfrak{m}$ such that $a_n + m_n = 1$. This implies $1 - a_n \in \mathfrak{m}^n$, hence $(1+\mathfrak{a}^n)\cap\mathfrak{m} \neq \emptyset$ for all $n \in \mathbb{N}$. Since $\{1+\mathfrak{a}^n : n \in \mathbb{N}\}$ is a neighborhood basis of 1 in $\mathfrak{a}$-topology, $1 \in \overline{\mathfrak{m}}$, the closure of $\mathfrak{m}$. This implies $\mathfrak{m}$ is not closed.

7. If $\hat{A}$ is faithfully flat over A, then for any finitely generated module M, $M \to \hat{A} \bigotimes_A M \cong \hat{M}$ is injective. In particular, this is true for $A/\mathfrak{m}$, where $\mathfrak{m}$ is any maximal ideal. Thus, $\ker(A/\mathfrak{m} \to \hat{A}/\hat{m}) = \bigcap_{n\geq 0}\mathfrak{a}^n(A/\mathfrak{m}) = 0$. Now suppose to get a contradiction that A is not Zariski ring; then there is a maximal ideal $\mathfrak{m}$ which doesn't contain $\mathfrak{a}$, thus $\exists a \in \mathfrak{a} - \mathfrak{m}$, thus $\mathfrak{a}(A/\mathfrak{m}) = A/\mathfrak{m}$ since $\overline{a} \neq 0$ and $A/\mathfrak{m}$ is a field, thus $\overline{a}$ is unit. Therefore, $\mathfrak{a}^n(A/\mathfrak{m}) = A/\mathfrak{m}$ for any $n \in \mathbb{N}$. This implies $\bigcap_{n\geq 0}\mathfrak{a}^n(A/\mathfrak{m}) = A/\mathfrak{m} \neq 0$, contradiction. Hence, A is Zariski ring.

   Conversely, if A is Zariski ring, then $\mathfrak{a} \subseteq \mathfrak{m}$, thus $\bigcap_{n\geq 0}\mathfrak{a}^n(A/\mathfrak{m}) = 0$, since $\mathfrak{a}$ acts on $A/\mathfrak{m}$ only trivially. Since $\ker(A/\mathfrak{m} \to \hat{A}/\hat{m}) = \bigcap_{n\geq 0}\mathfrak{a}^n(A/\mathfrak{m})$ comes from the construction of completion, we can say that $A/\mathfrak{m} \to \hat{A}/\hat{m}$ is injective. Thus, from the fact that $A/\mathfrak{m}$ is nonzero, $\hat{A}/\hat{m}$ is nonzero. This implies $\hat{m} = \mathfrak{m}^e$ is proper ideal of $\hat{A}$. By Exercise 3.16 iii), this implies that $\hat{A}$ is faithful flat A-algebra.

8. We claim that

   Claim LVI. *If $f \in B$ has nonzero constant term, then f is unit in B.*

   Proof. If $n = 1$, done by Exericse 1.5 i). For $n > 1$, suppose that f has nonzero constant term. Also, by mutliplying suitable constant, assume $f(0,\cdots,0) = 1$. Then, if f^{-1} exists, then $f^{-1} = \sum_k (1-f)^k$ by geometric series; letting $g = (1-f)$, $f^{-1} = (1-g)^{-1} = \sum_k g^k =$. Now, since $1-f$ has no constant term, $\sum_k(1-f)^k$ is well-defined; since for each coefficient of fixed degree, say d, is determined by finite partial sum $\sum_{k=0}^d(1-f)^k$, and there is only one constant term, which is $(1-f)^0 = 1$. This shows that f^{-1} exists.

   Now, to show $f^{-1} \in B$, we need to show that f^{-1} has also positive radius of convergence if f has. Suppose $f(0) = 1$. Then, since f has positive radius of convergence near 0, so does $g = f - 1$. Notes that since $f^{-1} = \frac{1}{1+g} = 1 - g + g^2 + \cdots$, and we know that the geometric series converges when $|g| < 1$. Hence, from $g(0) = 0$, let $r > 0$ such that g converges and $|g(z)| < 1$ for any $z \in B_r(0)$. Notes that this r is nonzero, since $g(0) = 0$ and continuity of g implies $\exists r_1 > 0$ such that $|g(z)| < 1$ for all $z \in B_{r_1}(0)$; also, by letting R be the convergence of radius, $\min(r_1, R) > 0$ guarantees that r is nonzero. Hence, f^{-1} converges when $z \in B_r(0)$. So it has positive radius of convergence. ☐

From this, B is a ring; to see it is local, notes that if f has nonzero constant in B, then $f^{-1} \in B$. Thus, by letting $\mathfrak{m} = \langle z_1, \cdots, z_n \rangle$ an ideal of B, $f \notin \mathfrak{m}$ is unit. Therefore, Proposition 1.6 shows that B is local ring with the unique maximal ideal $\mathfrak{m}$.

Notes that $A \subseteq B$ is obvious, since $g(0) \neq 0$ implies g has nonzero constant, thus invertible in C, which shows that $g^{-1} \in C$. Moreover, from g has infinite radius of convergence, so g^{-1} has positive radius of convegence, which implies both $g, g^{-1} \in B$. Also, any polynomial is in B. This implies $A \subseteq B$.

Moreover, B with completion is C, to see this, notes that $A \subseteq B \subseteq C$ with $\hat{A} = C = \hat{C}$ by Corollary 10.27. Observe that $A/(\mathfrak{m} \cap A)^n \cong B/\mathfrak{m}^n$ for any $n \in \mathbb{N}$, since image of power series in $B/\mathfrak{m}^n$ is truncation of power series up to degree n, which is polynomial, thus, the same as $A/(\mathfrak{m} \cap A)^n$. This implies A and B has the same inverse system with respect to $\mathfrak{m}$-adic topology, hence $\hat{A} = \hat{B} = C$.

Lastly, observe that A, B are Zariski ring; since A, B are local ring with $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is just Jacobson radical. By Exercise 10.7, $C = \hat{A} = \hat{B}$ is faithfully over A and B. Now notes that $A \to \hat{A}$ actually factor through $A \to B \to \hat{B}$, and we know that $A \to B \to \hat{B} = A \to \hat{A}$ is flat and $B \to \hat{B}$ is faithfully flat. Thus, by Exercise 3.17, $A \to B$ is flat.

9. We need to construct coprime monic polynomials g_k, h_k with degree r and $n-r$ such that $g_k h_k - f \in \mathfrak{m}^k A$. In $k = 1$, g_1, h_1 are given by the problem. Thus suppose we know g_{k-1} and h_{k-1} such that $\deg(g_{k-1}) = r, \deg(h_{k-1}) = n-r$ and $g_{k-1}h_{k-1} - f \in \mathfrak{m}^{k-1}A[x]$, and that their image $\overline{g_{k-1}}, \overline{h_{k-1}}$ are coprime in $(A/\mathfrak{m})[x]$, which is PID.

   First of all, by inductive hypothesis that $f(x) - g_{k-1}(x)h_{k-1}(x) \in \mathfrak{m}^{k-1}(A[x])$,

$$f(x) - g_{k-1}(x)h_{k-1}(x) := \sum_{p=0}^{n} c_p x^p$$

   for some $c_p \in \mathfrak{m}^{k-1}$. Now fix p such that $0 \leq p < n$. Then, since $(A/\mathfrak{m})[x]$ is a PID, we can apply Bezout' identity on $\overline{g_{k-1}}, \overline{h_{k-1}}$ to get $\overline{a_p}, \overline{b_p} \in (A/\mathfrak{m})[x]$ such that $\overline{a_p g_{k-1}} + \overline{b_p h_{k-1}} = \overline{x^p}$. Now by Euclidean lemma, there exists $q, r \in (A/\mathfrak{m})[x]$ such that $\overline{a_p} = \overline{h_{k-1}}q + r$ with $\deg r < \deg \overline{h_{k-1}} = n - r$. Then,

$$r\overline{g_{k-1}} + (q\overline{g_{k-1}} + \overline{b_p})\overline{h_{k-1}} = x^p$$

   Thus, by replacing $\overline{a_p}$ with r and $\overline{b_p}$ with $(q\overline{g_{k-1}} + \overline{b_p})$, we may assume that $\deg \overline{a_p} < n - r$. Then,

$$\deg \overline{b_p h_{k-1}} = \deg x^p - \overline{a_p g_{k-1}} < \deg f.$$

   Thus, $\deg \overline{b_p} < \deg \overline{g_{k-1}} = r$ Therefore, we may choose a_p and b_p such that for each $0 \leq p < n$, $\exists a_p, b_p \in A[x]$ such that $a_p g_{k-1} + b_p h_{k-1} - x^p \in \mathfrak{m}(A[x])$ with $\deg(a_p) < n - r, \deg(b_p) < r$.

   Thus, if we let $r_p(x) = a_p g_{k-1} + b_p h_{k-1} - x^p \in \mathfrak{m}(A[x])$, then $x^p = a_p g_{k-1} + b_p h_{k-1} - r_p(x)$, thus

$$f(x) - g_{k-1}(x)h_{k-1}(x) = \sum_{p=0}^{n} c_p x^p = \sum_{p=0}^{n} c_p(a_p(x)g_{k-1}(x) + b_p(x)h_{k-1}(x) - r_p(x)) \in \mathfrak{m}^{k-1}(A[x])$$

   From this, let $g_k = g_{k-1} + \sum_{p=0}^{n} c_p b_p(x), h_k = h_{k-1} + \sum_{p=0}^{n} c_p a_p(x)$. Then,

$$g_k h_k = g_{k-1}h_{k-1} + \sum_{p=0}^{n} c_p(a_p(x)g_{k-1}(x) + b_p(x)h_{k-1}(x)) + \left(\sum_{p=0}^{n} c_p b_p(x)\right)\left(\sum_{p=0}^{n} c_p a_p(x)\right).$$

   thus

$$f(x) - g_k h_k = \sum_{p=0}^{n} c_p r_p(x) + \left(\sum_{p=0}^{n} c_p b_p(x)\right)\left(\sum_{p=0}^{n} c_p a_p(x)\right) \equiv 0 \bmod \mathfrak{m}^k$$

   since $c_p \in \mathfrak{m}^{k-1}$ and $r_p(x) \in \mathfrak{m}(A[x])$ by construction, and $c_p \cdot c_{p'} \in \mathfrak{m}^{2k-2} \subseteq \mathfrak{m}^k$. Moreover, $g_k \equiv g_{k-1} \bmod \mathfrak{m}^{k-1}$ and $h_k \equiv h_{k-1} \bmod \mathfrak{m}^{k-1}$ since $c_p \in \mathfrak{m}^{k-1}$.

In summary, we just showed that for each $k \in \mathbb{N}$, $\exists g_k, h_k \in A[x]$ such that $f - g_k h_k \in \mathfrak{m}^k$ and $g_{k+1} \equiv g_k$ mod $\mathfrak{m}^k$ and $h_{k+1} \equiv h_k$ mod $\mathfrak{m}^k$. Thus, if we think about the map $\theta_{k+1} : A/\mathfrak{m}^{k+1} \to A/\mathfrak{m}^k$ consisting inverse system, then $\theta_{k+1}(\overline{g_{k+1}}) = \overline{g_k}$ since θ is just a map sending each coefficeint $\overline{c} = c + \mathfrak{m}^{k+1}$ of $\overline{g_{k+1}} \in (A/\mathfrak{m}^{k+1})$ to $c + \mathfrak{m}^k$, thus $\theta_{k+1}(\overline{g_{k+1}}) = \overline{g_{k+1}}$ mod $\mathfrak{m}^k \equiv \overline{g_k}$. Hence, if we suppose $g_k := \sum_{n=0}^{r} c_{k,n} x^n$, then $(c_{k,n})_{k \in \mathbb{N}}$ form a Cauchy sequence for each n, since for any $\mathfrak{m}^k$, a basic open neighborhood of 0, for any $k_1, k_2 > k$, we have $c_{k_1,n} - c_{k_2,n} \equiv c_{k,n} - c_{k,n}$ mod $\mathfrak{m}^k = 0$ mod $\mathfrak{m}^k$ implies that $c_{k_1,n} - c_{k_2,n} \in \mathfrak{m}^k$. Therefore, since A is complete, $\exists c_n = \lim_k c_k$ in A. Now let $g = \sum_{n=0}^{r} c_n x^n$. In the same way, define h. Then, $g \equiv g_k$ mod $\mathfrak{m}^k$, $h \equiv h_k$ mod $\mathfrak{m}^k$, thus $f - gh \equiv f - g_k h_k$ mod $\mathfrak{m}^k[x] \equiv 0$ mod $\mathfrak{m}^k[x]$ for all $k \in \mathbb{N}$. Thus, $f - gh \in \bigcap_{k \geq 0} \mathfrak{m}^k[x]$. Since A is complete, $\bigcap_{k \geq 0} \mathfrak{m}^k = 0$, which implies $f - gh = 0$.

10. (a) From definition of simple root, $\overline{f}$ is factorized as $\overline{f} = (\overline{x} - \alpha)\overline{h}(x)$. Let $\overline{g} = \overline{x} - \alpha$. By applying Hensel's lemma, we have $g, h \in A[x]$ such that $f = gh$. Since $\deg(\overline{g}) = 1$, $g = x - a$ for some $a \in A$. Thus, f has simple root $a \in A$ such that $a \equiv \alpha$ mod $\mathfrak{m}$.

    (b) To see this we need to find a solution of $x^2 - 2$ in the 7-adic completion of $\mathbb{Z}$. Moreover, if solution α exists, then $x^2 - 2$ splits into $(x - \alpha)(x + \alpha)$. Now if we let $A = \mathbb{Z}_7$, the completion of $\mathbb{Z}$ by (7), then $\mathfrak{m} = (7)$, thus $A/\mathfrak{m} = \mathbb{Z}_7/7\mathbb{Z}_7 \cong \mathbb{Z}/7\mathbb{Z}$ by Proposition 10.15. Now notes that $3^2 = 9 \equiv 2$ mod 7 is the solution of this equation, thus Hensel lemma lifts $\overline{3}$ to some element in $\mathbb{Z}_7$. Thus 2 is square in $\mathbb{Z}_7$

    (c) By thinking $A = k[[x]], \mathfrak{m} = (x)$, we have $k[x, y] \subseteq A[y]$. Then, $f(0, y) = \overline{f}$ in $(A/\mathfrak{m})[y]$. Thus, $f(0, y) = (y - a_0)h(y)$ induces $f = (y - a)h(x, y)$ in $A[x]$. Thus, f has a solution $a \in A = k[[x]]$. Thus, by letting $y(x) = a$, $f(x, y(x)) = (a - a)h(x, y(x)) = 0$.

11. Let A be the ring of germs of C^∞ functions of x at $x = 0$. Notes that f and g are in the same germ of 0 if and only if $\exists U$ an open neighborhood of 0 such that $f|_U = g|_U$. Also, $ev : A \to \mathbb{R}$ by $[f] \mapsto f(0)$ gives that $\mathfrak{m} = \{[f] : f(0) = 0\}$ is a maximal ideal. Also, if $[f]$ admits $f(0) \neq 0$, then by inverse function theorem f admits local inverse at 0, so $\exists [g]$ such that $[f][g] = [1]$. Thus, $[f]$ is unit if it is not in $\mathfrak{m}$. Thus A is local ring.

    First of all, we claim that

    $$\mathfrak{m}^k = \{[f] \in A : \frac{d^j}{dx}(f)(0) = 0 \text{ for all } 0 \leq j < k\}.$$

    To see this, if $f_1 \cdots f_k$ are product of elements in $\mathfrak{m}$, then by product rule of differentiation, each term occurs at j-th derivatives contain at least one of f_i, thus 0 when $x = 0$. Conversely, if f is in the right hand side, then by Taylor's theorem, for any open neighborhood U of 0,

    $$f(x) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(0) x^j + g_n(x) x^n = g_n(x) x^n$$

    where $g_n \in C^\infty(U)$ and $g_n(0) = \frac{1}{k!} f^{(k)}(0)$. Thus $f \in ([x^n]) \subseteq \mathfrak{m}^k$. Done.

    And from this argument, we see that $\mathfrak{m}^k \subseteq \{[f] \in A : \frac{d^j}{dx}(f)(0) = 0 \text{ for all } 0 \leq j < k\} \subseteq ([x^n]) \subseteq \mathfrak{m}^k$. Hence, $\mathfrak{m}^k = ([x^n])$

    Now to understand $\hat{A}$, we claim that $A/\mathfrak{m}^k \cong \mathbb{R}[x]/(x^k)$ for all k. Notes that $R[x] \mapsto A$ sends (x^n) to $([x^n])$, so $R[x]/(x^k) \cong A/\mathfrak{m}^k = A/([x^k]$ by the first isomorphism theorem. Also, This commutes with the map $\theta_{k+1} : A/\mathfrak{m}^{k+1} \to A/\mathfrak{m}^k$ and $\theta'_{k+1} : \mathbb{R}[x]/(x^{k+1}) \to \mathbb{R}[x]/(x^k)$ since both of the works by picking representation in $R[x]$ (or A) and mod out by (x^k) (or $\mathfrak{m}^k$). Thus, $\hat{A} \cong \widehat{\mathbb{R}[x]} = \mathbb{R}[[x]]$. Since $\mathbb{R}$ is field, so it is Noetherian, hence by Corollary 10.27, $\hat{A}$ is Noetherian.

    Then, $\hat{A}$ is finitely generated A-module since $A \to \hat{A} \cong \mathbb{R}[[x]]$ is sending f to its Taylor expansion. (One can actually use this map to showing that $A/\mathfrak{m}^k \cong \mathbb{R}[x]/(x^k)$) Hence, by Borel's theorem, $A \to \hat{A}$ is surjective. Thus, as a module, $\hat{A}$ is finitely generated module since image of $([1])$ generates all $\hat{A}$.

    However, A is not Noetherian, since $\bigcap_{n \in \mathbb{N}}([x^n]) \neq 0$; think e^{-1/x^2}. Since its Taylor expansion at 0 is 0, it is inside of $([x^n])$ for all n. Thus, by Corollary 10.18, A is not Noetherian.

12. Notes that $A \to A[x_1, \cdots, x_n]$ is flat by applying Exercise 2.5 and 2.8 ii) n times. Now, notes that $A[x_1, \cdots, x_n] \to A[[x_1, \cdots, x_n]]$ is faithfully flat, since Exercise 1.5 v) shows $\mathrm{Spec}(A[[x_1, \cdots, x_n]]) \to \mathrm{Spec}(A[x_1, \cdots, x_n])$ is surjective, which is one of equivalent conditions of faithfully flat. Now apply Exercise 2.8 ii) on $A \to A[x_1, \cdots, x_n] \to A[[x_1, \cdots, x_n]]$ to get $A \to A[[x_1, \cdots, x_n]]$ is flat.

11 Dimension Theory

In the proof of Theorem 11.1, L is annihilated by x_s since $L_n = M_{n+k_s}/x_sM_n$; thus if we multiply x_s on $\overline{m} \in L_n$, then $x_s \cdot m \in x_sM_{n+2k_s}$, thus $\overline{x_sm} = 0 \in M_{n+2k_s}/x_sM_{n+k_s}$. Also, to get (2), by summing the above line, you get

$$P(K,t) - P(M,t) + \sum_{n=k_s}^{\infty} \lambda(M_n)t^{n-k_s} = \sum_{n=k_s}^{\infty} \lambda(L_n)t^{n-k_s}.$$

Multiply t^{k_s}, then you can get

$$t^{k_s}P(K,t) - t^{k_s}P(M,t) + \sum_{n=k_s}^{\infty} \lambda(M_n)t^{n} = \sum_{n=k_s}^{\infty} \lambda(L_n)t^{n}.$$

Then, $\sum_{n=k_s}^{\infty} \lambda(M_n)t^n = P(M,t) - g_1(t)$ and $\sum_{n=k_s}^{\infty} \lambda(L_n)t^n = P(L,t) - g_2(t)$. Thus, by letting $g = g_2(t) - g_1(t)$,

$$t^{k_s}P(K,t) - t^{k_s}P(M,t) + P(M,t) = P(L,t) + g(t).$$

Now by reordering you can get (2). Now inductive hypothesis shows that $P(L,t), P(K,t)$ are of given form, thus so is $P(M,t)$.

In the proof of Corollary 11.2, $(1-t)^{-d} = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} t^k$ is from following; since $(1-t)^{-d} = \prod_{i=1}^{d}(1 + t + t^2 + \cdots)$, thus the coefficient of t^k is the number of ordered d-partitions of $[k]$. Actually, it is the same as giving $d-1$ bars on the sequence of same k balls to make d groups. Thus, it is the same as choosing $d-1$ positions in a row of $d-1+k$ positions and assign bars on the chosen positions and put balls on the other positions. Thus, we get $\binom{d+k-1}{d-1}$.

Now, for $\lambda(M_n)$ if $n \geq N$, then since it is coefficient of t^n with $n \geq N$ in $f(t) \cdot (1-t)^{-d}$, it is sum of $a_kt^k \times \binom{d+n-k-1}{d-1}t^{n-k}$ for $k = 0, 1, \cdots, N$. Done.

Also notes that d in corollary 11.2 is the order of the pole.

To get $d(L) = d(M)-1$, notes that in given situation, $L = M/xM$ and by applying λ on $0 \to M_0 \to M_1 \to L_1 \to 0$ you get $\lambda(L_1) = \lambda(M_1) - \lambda(M_0)$. And, (2) gives $(1-t)P(M,1) + \lambda(M_0) = P(L,1) + \lambda(L_0) = P(L,1)$ since L_0 does not exists. Thus, $P(L,1) - \lambda(M_0) = (1-t)P(M_1)$. Since $\lambda(M_0) \in \mathbb{Z}$, the order of the pole of $P(L,1)$ is 1 less than that of M. This shows $d(L) = d(M) - 1$.

In the example in p.118, length of A_n can be obtained by a chain of submodule, whose n-th module is generated by $\binom{s+n-1}{s-1} - n$ monomials.

In the proof of Proposition 11.4, to see that $G(A)$ is Noetherian an $G(M)$ is finitely generated graded $G(A)$-module use Proposition 10.22. Also, $A/\mathfrak{q}$ is Artin, since Corollary 7.16 shows that $\exists n \in \mathbb{N}$ such that $\mathfrak{m}^n \subseteq \mathfrak{q}$, thus $\overline{\mathfrak{m}}$ in $A/\mathfrak{q}$ is nilpotent. Now apply Proposition 8.6. Also, a finitely generated module M over Artinian local ring A is artinian, since $0 \to \ker(A^n \to M) \to A^n \to M \to 0$ is exact sequence of A-module, and A^n is also Artinian by Theorem 8.7. Now apply Proposition 6.3. And M/M_n is of finite length since each M_{n-k}/M_n with $k \geq 1$ has finite length; when $k = 1$, done. Suppose M_{n-k+1}/M_n has finite length. Then, it has a short exact sequence

$$0 \to M_{n-k+1}/M_n \to M_{n-k}/M_n \to M_{n-k}/M_{n-k+1} \to 0$$

Since each M_{n-k+1}/M_n and M_{n-k}/M_{n-k+1} have the composition series, thus Proposition 6.8 says that it satisfies both chain condition. By Proposition 6.3, so does M_{n-k}/M_n. Now $l(\cdot)$ is an additive map on module with finite length, the exact sequence induces

$$l(M_{n-k}/M_n) - l(M_{n-k+1}/M_n) = l(M_{n-k}/M_{n-k+1}).$$

Thus this finite difference implies $l(M/M_n) = \sum_{r=1}^{n} l(M_{r-1}/M_r)$. Also, as a polynomial notes that l_k is a polynomial in n of degree $\leq s-1$ distinct with $f(n)$, by thinking it as a coefficient of power series. Thus, sum l_n is a polynomial $g(n)$ of degree $\leq s$ for all large n. Lastly, to get $\lim_{n\to\infty} g(n)/\tilde{g}(n) = 1$, apply limit on $g(n)/g(n+n_0) \leq g(n)/\tilde{g}(n) \leq \tilde{g}(n+n_0)/\tilde{g}(n)$.

In proof of Proposition 11.6, containment affects on the size of polynomial since it is just coefficient as a length. Thus, by taking $n \to \infty$ on $1 \leq \xi_{\mathfrak{q}}(n)/\xi_{\mathfrak{m}}(n) \leq \xi_{\mathfrak{m}}(rn)/\xi_{\mathfrak{m}}(n)$, we get $\lim_{n\to\infty} \xi_{\mathfrak{m}}(rn)/\xi_{\mathfrak{m}}(n) = r^{\deg \xi_{\mathfrak{m}}(n)} > 0$ when $r \geq 1$. Hence limit of $\xi_{\mathfrak{q}}(n)/\xi_{\mathfrak{m}}(n)$ converges; this implies $\xi_{\mathfrak{q}}(n)$ and $\xi_{\mathfrak{m}}(n)$ have the same degree.

To check $d(A) = d(G_{\mathfrak{m}}(A))$, I refer [11][11.2]. Let old d in [3][p.117] is d_0 and new d in [3][p.119] be d_v. Then it is the problem of checking $d_v(A) := d_0(G_{\mathfrak{m}}(A))$. Notes that $G_{\mathfrak{m}}(A)$ is Noetherian graded ring since A is Noetherian local and Proposition 10.22 (i). If we assume $M = A$ in Proposition 11.4, $l_n = l(A/\mathfrak{m}^n)$. Corollary 11.4(1) shows that $l_n = \sum_{r=1}^{n} l(\mathfrak{m}^r/\mathfrak{m}^{r+1})$. And definition of $d_0(G_{\mathfrak{m}}(A))$ is a pole of $P(G_{\mathfrak{m}}(A),1) = \sum_{r=1}^{\infty} l(\mathfrak{m}^r/\mathfrak{m}^{r+1})t^r$. Let $Q(A,1) := \sum_{r=1}^{\infty} l_r t^r$. Then, $P(G_{\mathfrak{m}}(A),1) \cdot \sum_{q}^{\infty} t^q = Q(A,1)$, i.e., $P(G_{\mathfrak{m}}(A),1) \cdot \frac{1}{1-t} = Q(A,1)$. Hence, $d_0(G_{\mathfrak{m}}(A)) + 1 = d_0(A)$ by definition of the pole. By applying Corollary 11.2 on $Q(A,1)$, we can get $d_0(A) - 1 = d_v(A)$. Thus,

$$d_0(G_{\mathfrak{m}}(A)) = d_0(A) - 1 = d_v(A).$$

Or, you can see that

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to A/\mathfrak{m}^{n+1} \to A/\mathfrak{m} \to 0$$

This is exact by third isomorphism theorem. (Take quotient and apply it.) Then, by multiplying t, we can get

$$(1-t)Q(A,1) = tP(G_{\mathfrak{m}}(A),1) + l(A/\mathfrak{m})$$

Hence, $d_0(G_{\mathfrak{m}}) = d_0(A) - 1$. Now we can get the same answer.

In the proof of 11.8, the exactness of short exact sequence can be checked by showing $(M/\mathfrak{q}^n M)/(xM/\mathfrak{q}^n M) \cong (M/xM)/\mathfrak{q}^n(M/xM)$ directly. The map is just sending representation to representation. However, I didn't know whether $A/\mathfrak{q}^n A$ is flat or not.

For the example in [3][p.121], notes that $G_{\mathfrak{m}}(A)$ is a polynomial ring over $A/\mathfrak{m}$ with n indeterminates. Thus, length of each module is just the number of monomials with particular degree. This leads to the Poincare series in the book. Thus, $d_0(G_{\mathfrak{m}}(A)) = n \implies d_v(A) = n \implies \dim A = n$ by Theorem 11.14 $\implies \dim A_{\mathfrak{m}} = n$ since any prime in A contained in $\mathfrak{m}$.

In corollary 11.15, since $\mathfrak{m}$ is generated by $\dim A = \delta(A)$ number of elements by the dimension theorem $s \geq \dim A$.

In the proof of Proposition 11.20, d is old d, i.e., d_0 with $\lambda = l$. Thus, $d(G_{\mathfrak{q}}(A)) \leq d((A/\mathfrak{q})[t_1, \cdots, t_d]/(\overline{f}))$ comes from the fact that $(A/\mathfrak{q})[t_1, \cdots, t_d]/(\overline{f}) \to G_{\mathfrak{q}}(A)$ is still surjective since $\overline{f} \in \ker(\alpha)$, thus preimage of chain of submodules in each graded summand of the latter is also chain in former. The rest is straighforward, and the last equation $d(G_{\mathfrak{q}}(A)) = d$ is actually $d_o(G_{\mathfrak{q}}(A)) = d_o(G_{\mathfrak{m}}(A)) = d_v(A) = d$ where first equality comes from the previous argument that it doesn't matter on the pole whether use $\mathfrak{m}$-primary ideal or $\mathfrak{m}$ itself by Proposition 111.6. And the second equaltiy is just definition of d_v, and the third equality comes from the example in [3][p.121].

In the proof of Corollary 11.21, $f_s(x_1, \cdots, x_d) = -$higher terms, thus it is in $\mathfrak{q}^{s+1}$. That's why we can apply 11.20 in this case.

In the proof of theorem 11.22, iii) $\to$ i), notes that generators $x_1, \cdots, x_d$ of $\mathfrak{m}$ (which is clearly $\mathfrak{m}$-primary ideal) is a system of parameter since $\dim A = d$. Thus, by Corollary 11.21, they are algebraically independent over k. Thus, the epimorphism α constructed in Proposition 11.20 must have no kernel elements except 0.

For the statement that regular local ring is integrally closed, see [12][Proposition 2.2.3].

Notes that the strong Nullstellensatz (Exercise 7.14) implies that for any ideal $\mathfrak{a}$ of $k[x_1, \cdots, x_n]$, $I(V(\mathfrak{a})) = r(\mathfrak{a})$. In case of when $\mathfrak{a} = \mathfrak{m}$, then $V(\mathfrak{m})$ is a point of k^n. Thus it gives a bijection between k^n and maximal ideals of $k[x_1, \cdots, x_n]$.

In the proof of Theorem 11.25, notes that $\dim V$ is actually the number of algebraically independents of $k(V)$. And $k(V)$, a field of fraction of $A(V)$, contains $A(V)_{\mathfrak{m}}$ as a subring. Thus, by Corollary 11.21, since $A(V)_{\mathfrak{m}}$ contains k and has a system of parameters, which is a generators of $\mathfrak{m}A(V)_{\mathfrak{m}}$, then they are algebraically independent over k. Thus, the number of algebraically independents elements of $k(V)$ has at least the number of the system of parameters of $A(V)_{\mathfrak{m}}$, which is $\dim A(V)_{\mathfrak{m}}$, since $A(V)_{\mathfrak{m}} \subseteq k(V)$. Hence, $\dim V \geq \dim A(V)_{\mathfrak{m}}$.

In the proof of Lemma 11.26, integally closed means integrally closed in the field of fraction.

1. Let $P = (p_1, \cdots, p_n)$. Since k is algebraically closed field, $f = \sum_{i=1}^{n}(x_i - p_i)g_i$ for some $g_i \in k[x_1, \cdots, x_n]$. Hence,
$$\frac{\partial f}{\partial x_i} = g_i + (x_i - p_i)\frac{\partial g_i}{\partial x_i} + \sum_{j \neq i}^{n}(x_j - p_j)\frac{\partial g_j}{\partial x_j}.$$
Thus, f is singular at P if and only if $g_i \in \mathfrak{m}_P^2$ if and only if $f \in \mathfrak{m}_P^2$. Now notes that $\mathfrak{m}$ is image of $\mathfrak{m}_P$ in $A = k[x_1, \cdots, x_n]/(f)$. Thus, image of $k[x_1, \cdots, x_n] - \mathfrak{m}_P$ is just $A - \mathfrak{m}$. Hence,
$$k[x_1, \cdots, x_n]_{\mathfrak{m}_P}/(f)_{\mathfrak{m}_P} \underbrace{\cong}_{Corollary 3.4iii)} (k[x_1, \cdots, x_n]/(f))_{\mathfrak{m}_P} \underbrace{\cong}_{Exercise 3.4} (k[x_1, \cdots, x_n]/(f))_{\mathfrak{m}} = A_{\mathfrak{m}}.$$
Also, $\mathfrak{m}^2$ can be denoted as an image of $\mathfrak{m}_P^2$, thus $\mathfrak{m}^2 = (\mathfrak{m}_P^2 + (f))/(f)$. Thus, $\mathfrak{m}/\mathfrak{m}^2 \cong (\mathfrak{m}_P/(f))/(\mathfrak{m}_P^2 + (f))/(f) \cong \mathfrak{m}_P/(\mathfrak{m}_P^2 + (f))$ where the last equality comes from third isomorphism theorem. Now, f is singular if and only if $f \in \mathfrak{m}_P^2$ if and only if $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{m}_P/(\mathfrak{m}_P^2)$ if and only if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim_k(\mathfrak{m}_P/(\mathfrak{m}_P^2)) = n$. Thus, when f is singular, $\dim A_{\mathfrak{m}} = \dim A = \dim k[x_1, \cdots, x_n]/(f) = n - 1 \neq n = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. So A is not regular. If f is nonsingular, then $\mathfrak{m}_P^2 + (f)$ is strictly greater than $\mathfrak{m}_P$, hence there is $v \in \mathfrak{m}_P$ which is nonzero in $\mathfrak{m}_P/\mathfrak{m}_P^2$ but zero in $\mathfrak{m}_P/(\mathfrak{m}_P^2 + (f))$. Since $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a vector space, there is a basis containing v. This implies that as a vector space $\mathfrak{m}_P/(\mathfrak{m}_P^2 + (f))$ has dimension less than that of $\mathfrak{m}_P/\mathfrak{m}_P^2$. Thus, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq n - 1$. Now apply Corollary 11.5 on $A_{\mathfrak{m}}$ we get $n - 1 = \dim A_{\mathfrak{m}} \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq n - 1$. This shows $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = n - 1$. Hence $A_{\mathfrak{m}}$ is regular.

2. This homomorphism is lifting of $k[t_1, \cdots, t_d]/(t_1, \cdots, t_d)^n \to A/(x_1, \cdots, x_d)^n$ as $t_i \mapsto x_i$ into their completion. This map is injective for any n, since $p(x) = 0$ implies the least degree monomial of $p(x)$ is at least n, thus the least degree monomial of $p(t)$ is at least n, hence $p(t) = 0$. Thus, by Proposition 10.2, the map between completion is injective. Now all we need to show is that $(x_1, \cdots, x_d)$-adic completion is equal to $\mathfrak{m}$-adic completion. Since $(x_1, \cdots, x_d)$ is $\mathfrak{m}$-primary and A is Noetherian, Corollary 7.16 shows that $\mathfrak{m}^n \subseteq (x_1, \cdots, x_d) \subseteq \mathfrak{m}$ for some $n > 0$. Thus, as a neighborhood base, $(x_1, \cdots, x_d)$-adic basis and $\mathfrak{m}$-adic basis has the same open set. Hence, $(x_1, \cdots, x_d)$-adic completion is equal to $\mathfrak{m}$-adic completion. Hence the given homomorphism is injective.

   Now we may think that A is $k[[t_1, \cdots, t_d]]$ module. $\{(x_1, \cdots, x_d)^n\}_{n \in \mathbb{N}}$ is $(t_1, \cdots, t_d)$-filtration of A. Moreover, since A is $\mathfrak{m}$-adic completion, which is equivalent to $\mathfrak{q}$-adic completion, thus A is Hausdorff in its filtration topology. Now let $\mathfrak{a} = (x_1, \cdots, x_d)$. Notes that $k[[t_1, \cdots, t_d]] \to A$ is injective. By below claim, $k[[t_1, \cdots, t_d]]/(t_1, \cdots, t_d) \to A/(x_1, \cdots, x_d)$ is injective. Hence, $G_{(t_1, \cdots, t_d)}(k[[t_1, \cdots, t_d]])$ has a natural map on $G_{(x_1, \cdots, x_d)}(A)$ since their zeroth component is isomorphism and their other components are generated by correspondence $\overline{t_i} \mapsto \overline{x_i}$. Thus, $G_{(x_1, \cdots, x_d)}(A)$ is generated by 1 in $G_{(t_1, \cdots, t_d)}(k[[t_1, \cdots, t_d]])$ as a $G_{(t_1, \cdots, t_d)}(k[[t_1, \cdots, t_d]])$-module. So it is finitely generated $G_{(t_1, \cdots, t_d)}(k[[t_1, \cdots, t_d]])$-module. Thus, By Proposition 10.24, A is finitely generated $k[[t_1, \cdots, t_d]]$-module.

   Claim LVII. *Let $\alpha : A \to B$ be an injective ring homomorphism. Let I be an ideal of A. Then, $\alpha' : A/I \to B/I^e$ is injective ring homomorphism.*

   Proof. Suppose that $\alpha'(\overline{a}) = \overline{\alpha(a)} = 0$. Then $\alpha(a) \in I^e \cap \alpha(B) = \alpha(I)$. Thus $a \in I$. Hence $\overline{a} = 0$. □

3. From Noether normalization, there is a ring $B = k[x_1, \cdots, x_d]$ contained in $A(V)$ such that $d = \dim V$ and $A(V)$ is integral over B. By Lemma 11.26,
$$\dim B_{\mathfrak{n}} = \dim A(V)_{\mathfrak{m}}$$
Also by the hint $\overline{k}[x_1, \cdots, x_d]$ is integral over $k[x_1, \cdots, x_d]$. Thus, by applying Lemma 11.26,
$$\dim B_{\mathfrak{n}} = \dim \overline{k}[x_1, \cdots, x_d]_{\mathfrak{q}}.$$

By picking $\mathfrak{m}$ and $\mathfrak{q}$ such that $\mathfrak{m} \cap B = \mathfrak{n} = \mathfrak{q} \cap \overline{k}[x_1, \cdots, x_d]$, and from the theorem 11.25 implies that $\dim \overline{k}[x_1, \cdots, x_d]_{\mathfrak{q}} = d$ where d is a dimension of variety V over $\overline{k}$ for all maximal ideal $\mathfrak{q}$ of $\overline{k}[x_1, \cdots, x_d]$. Hence, $\dim B_{\mathfrak{n}} = d$ for all maximal ideal $\mathfrak{n}$. Since B was chosen as $d = \dim V$ where $\dim V$ is dimension of variety V over k, this implies the local dimension of V at any point is equal to $\dim V$.

4. Just notes that $\mathfrak{p}_i = (x_{m_i+1}, \cdots, x_{m_{i+1}})$. Now we claim that maximal ideals of $S^{-1}A$ are $\mathfrak{p}_i$s. Now if $\mathfrak{a}$ is an ideal of A does not meet S, then $\mathfrak{a}$ is inside of union of $\mathfrak{p}_i$s. Actually, since A is Noetherian, $\mathfrak{a}$ is finitely generated, and each generator of $\mathfrak{a}$ contains finitely many indeterminates. Hence $\mathfrak{a}$ is inside of finite union of $\mathfrak{p}_i$s. Thus, Proposition 1.11 i) shows that $\mathfrak{a}$ is contained in $\mathfrak{p}_i$. Thus by Proposition 3.11 i), all maximal ideals are $S^{-1}\mathfrak{p}_i$. Now, to use Exercise 7.9, we claim that $S^{-1}A_{S^{-1}\mathfrak{p}_i}$ is Noetherian. To see this, define a map
$$(S^{-1}A)_{S^{-1}\mathfrak{p}_i} \to A_{\mathfrak{p}_i} \text{ by } (a/b)/(c/d) \mapsto ad/bc.$$
Then, it is well-defined, since if $(a/b)/(c/d) = (e/f)/(g/h)$, then $\exists t/q \in S^{-1}A - S^{-1}\mathfrak{p}_i$ such that $tag/qbh = tce/qdf$, which implies $\exists l \in S$ such that $ltagqdf - lqbhtce = 0$. This implies $ltq(agdf - bhce) = 0$. Since $ltq \in S$, thus $ltq \in A \setminus \mathfrak{p}_i$. Hence, $ad/bc = eh/gf$ in $A_{\mathfrak{p}_i}$. Now surjectivity is clear; for any $a/b \in A_{\mathfrak{p}_i}$, send $(a/1)/(b/1)$. Injectivity is clear since $ad/bc = 0/1$ implies there is $t \in A - \mathfrak{p}_1$ such that $tad = 0$. Since $td \neq 0$ with A is integral domain, $a = 0$. Thus, $(0/b)/(c/d)$ is preimage of it, and notes that this is the same as $(0/1)/(1/1)$.

   Hence, this map is isomorphism as a ring. Thus, each $S^{-1}A_{S^{-1}\mathfrak{p}_i}$ is Noetherian since $A_{\mathfrak{p}_i}$ is Noetherian for any i by Corollary 7.4. Thus by applying exercise 7.9, $S^{-1}A$ is Noetherian. Now notes that each $S^{-1}p_i$ has height equal to $m_{i+1} - m_i$, which increases when $i \to \infty$. Thus, $\dim S^{-1}A = \infty$.

5. Suppose that γ is a map sending an A_0-module M to its class in $K(A_0)$. Then, for any homomorphism $\lambda_0 : K(A_0) \to \mathbb{Z}$ let $P(M,t) = \sum_{n=0}^{\infty} \lambda_0((M_n))t^n$. Then, $P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}$. Notes that proof is exactly same; since the construction of $K(A)$ implies that applying λ_0 on exact sequence is 0.

6. Follow hints; let $f : A \to A[x]$ be the embedding and consider the fiber of $f^* : \mathrm{Spec}(A[x]) \to \mathrm{Spec}(A)$ over a prime ideal $\mathfrak{p}$ of A. By Exercise 3.21 iv), $f^{*,-1}(\mathfrak{p}) \cong \mathrm{Spec}(k \bigotimes_A A[x])$ where k is the residue field of the local ring $A_{\mathfrak{p}}$. Now notes that $k \bigotimes_A A[x] \cong k[x]$ since $q \otimes ax^t = a(q \otimes x^t) = (\overline{a}q \otimes x^t)$ where $\overline{a}$ is image of a on k. Also, $\dim k[x] = 1$ since $k[x]$ is PID. Thus, if A has length r chain consisting of $r+1$ prime ideals, then $A[x]$ has at most length $2r+1$ chain consisting of $2r+2$ ideals. This shows $\dim A[x] \leq 1 + 2\dim A$.

   Conversely, if A has length r chain consisting of $r+1$ prime ideals, then Exercise 4.7 ii) implies that $A[x]$ has at least length r chain induced by chain of A. Moreover, let $\mathfrak{p}_r$ be the top elements of the chain of A. Then, $\mathfrak{p}_r + (x)$ is prime in $A[x]$; to see this, let $A[x] \to A \to A/\mathfrak{p}_r$. Then, this is surjective map. And kernel is $\mathfrak{p}_r + (x)$. Since $A/\mathfrak{p}_r$ is integral domain, so $\mathfrak{p}_r + (x)$ is prime. This implies $\dim A + 1 \leq \dim A[x]$.

7. Follow hints; We already know that $\dim A[x] \geq 1 + \dim A$. Thus we should show the other inequality. Let $\mathfrak{p}$ be a prime ideal of height $\mathfrak{m}$ in A. Then there exists $a_1, \cdots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is minimal prime ideal belonging the ideal $\mathfrak{a} = (a_1, \cdots, a_m)$. To see this, apply dimension theorem on $A_{\mathfrak{p}}$, which shows that there is $\mathfrak{p}A_{\mathfrak{p}}$-primary ideal with m generators. By Exercise 4.7 v), $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$, thus height $\mathfrak{p}[x] \leq \mathfrak{m}$ by Corollary 11.16.

   Now, let $\mathfrak{p}$ be a prime ideal of $A[x]$. Then, $\mathfrak{p}^c$ is prime ideal of A. If $\mathfrak{p}^c$ has height 0, then $\mathfrak{p}^c = 0$. Thus, from the fact that $f^{*,-1}(0)$ is set with two elements and one element is zero ideal of $A[x]$, $\mathfrak{p}$ is minimal ideal belong to 0; otherwise there exists another prime ideal between $\mathfrak{p}$ and 0, and this should be in $f^{*,-1}(0)$, contradiction. This shows height of $\mathfrak{p} \leq$ that of $\mathfrak{p}^c$+1 when $\mathfrak{p}^c$ has height 0.

   Then, suppose this holds when height of $\mathfrak{p}^c$ is m. Then, for any prime ideal $\mathfrak{q}$ strictly contained in $\mathfrak{p}$, height of $\mathfrak{q}^c$ is less than or equal to m. Now notes that $\mathfrak{p}^c[x]$ is a subset of $\mathfrak{p}$ such that both are in $f^{*,-1}(\mathfrak{p})$. If $\mathfrak{p}^c[x] = \mathfrak{p}$, then height of $\mathfrak{p}$ is less than height of $\mathfrak{p}^c + 1$. Done. Otherwise, there is no prime ideal between $\mathfrak{p}^c[x]$ and $\mathfrak{p}$ since $f^{*,-1}(\mathfrak{p}^c)$ has cardinality 2. In this case, $\mathfrak{p}^c[x]$ is strictly less than $\mathfrak{p}$, thus height m. Hence, $\mathfrak{p}$ has height less than $m+1$. This shows that every prime ideal of $A[x]$ has height less than 1+ height of its contraction. Hence, $\dim A[x] \leq 1 + \dim A$.

Now by Exercise 11.6, this shows $\dim A[x] = 1 + \dim A$. In case of polynomial ring with multiple variable, notes that Hilbert basis theorem shows that each polynomial subring is Noetherian. So apply inductively the above argument.

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